# 7. The Stability Theorem

Vin de Silva
Pomona College

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# The Persistence Stability Theorem

**Stability Theorem (Cohen-Steiner, Edelsbrunner, Harer 2007)**

The following transformation is Lipschitz:

data ⟶ persistence diagram

Small changes in the data lead to small changes in the diagram.

**Factorization**

data ⟶ persistence module ⟶ persistence diagram

**Geometric Stability**

data ⟶ persistence module is Lipschitz (in various settings)

**Algebraic Stability (Chazal, Cohen-Steiner, Glisse, Guibas, Oudot 2009)**

persistence module ⟶ persistence diagram is 1-Lipschitz

# Data

## Simplex stream

Let $S$ be a finite simplicial complex.

Let $f : S \to \mathbb{R}$ be a function with the property that

$$f^{-1}(-\infty, t]$$

is a subcomplex of $S$ for every $t \in \mathbb{R}$.

## Processing

Run the persistence algorithm on the sequence of simplices ordered by $f$-value.

## Metric

Two such functions $f, g : S \to \mathbb{R}$ are compared by the supremum norm:

$$\|f - g\|_\infty = \max_{\sigma \in S} |f(\sigma) - g(\sigma)|$$

# Data

## Sublevelset filtration

Let $M$ be a compact smooth manifold.

Let $f : M \to \mathbb{R}$ be a Morse function (critical points nondegenerate)

## Processing

Compute the persistence diagram of

$$\mathrm{H}_*(f^{-1}(-\infty, a_0]) \longrightarrow \dots \longrightarrow \mathrm{H}_*(f^{-1}(-\infty, a_n])$$

where $a_0 < \cdots < a_n$ are the critical values.

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Two such functions $f, g : M \to \mathbb{R}$ are compared by the supremum norm:

$$\|f - g\|_\infty = \max_{x \in M} |f(x) - g(x)|$$

# Data

## Sublevelset filtration

Let $P$ be a compact polyhedron ($\Leftrightarrow$ realization of a simplicial complex).

Let $f : P \to \mathbb{R}$ be a piecewise-linear map.

## Processing

Compute the persistence diagram of

$$\mathrm{H}_*(f^{-1}(-\infty, a_0]) \longrightarrow \ldots \longrightarrow \mathrm{H}_*(f^{-1}(-\infty, a_n])$$

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Two such functions $f, g : P \to \mathbb{R}$ are compared by the supremum norm:

$$\|f - g\|_\infty = \max_{x \in P} |f(x) - g(x)|$$

# Data

## Metric data (fixed set)

Let $X$ be a finite set.

Let $d : X \times X \to \mathbb{R}$ be a metric.

## Processing

Compute the persistence diagram of

$$\mathrm{H}_*\big(\mathrm{VR}((X,d),r_0)\big) \longrightarrow \dots \longrightarrow \mathrm{H}_*\big(\mathrm{VR}((X,d),r_n)\big)$$

where $r_0 < \cdots < r_n$ are the values taken by the metric.

## Metric

Two such metrics $d, e : X \times X \to \mathbb{R}$ are compared by the supremum norm:

$$\|d - e\|_\infty = \max_{x,y \in P} |d(x,y) - e(x,y)|$$

# Data

## Metric data (subsets of a fixed space)

Let $(Z, d)$ be a metric space.

Let $X \subseteq Z$ be a finite subset.

## Processing

Compute the persistence diagram of

$$\mathrm{H}_*\big(\mathsf{VR}((X,d),r_0)\big) \longrightarrow \dots \longrightarrow \mathrm{H}_*\big(\mathsf{VR}((X,d),r_n)\big)$$

where $r_0 < \dots < r_n$ are the values taken by the metric restricted to $X$.

## Metric

Two such subsets $X, Y \subseteq Z$ are compared by the **Hausdorff distance**:

$$d_{\mathrm{H}}(X,Y) = \max\left[\left(\max_{x \in X} d(x,Y)\right),\ \left(\max_{y \in Y} d(y,X)\right)\right]$$

# Data

## Metric data (general)

Let $(X, d)$ be a finite metric space.

## Processing

Compute the persistence diagram of

$$\mathrm{H}_*\big(\mathsf{VR}((X,d), r_0)\big) \longrightarrow \dots \longrightarrow \mathrm{H}_*\big(\mathsf{VR}((X,d), r_n)\big)$$

where $r_0 < \cdots < r_n$ are the values taken by the metric.

## Metric

Two metric spaces $(X, d)$, $(Y, e)$ are compared using the **Gromov–Hausdorff** distance:

$$d_{\mathrm{GH}}((X,d),(Y,e)) = \inf_{Z,f,g} \big[ d_{\mathrm{H}}(f(X), g(Y)) \big]$$

The min is taken over all metric spaces $Z$ and isometric embeddings $f : X \to Z$, $g : Y \to Z$.

# Persistence modules

## Persistence modules indexed by $\mathbb{R}$ (Chazal et al. 2009)

For the stability theorem, we need to work with real index values.

A **persistence module indexed by** $\mathbb{R}$ is specified by the following data:

- A vector space $V_t$ for every $t \in \mathbb{R}$.
- A linear map $v_t^s : V_s \to V_t$ whenever $s \leq t$.

We require $v_t^t = 1_{V_t}$ for all $t$, and $v_u^t v_t^s = v_u^s$ whenever $s \leq t \leq u$.

Henceforth, all persistence modules will be indexed by $\mathbb{R}$ unless stated otherwise.

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### Tameness

A persistence module is of **finite type** if there is a finite set of critical values

$$a_0 < a_1 < \cdots < a_n$$

such that

- $V_t = 0$ for all $t < a_0$; and
- $V_t$ is constant, and finite dimensional, over each interval $[a_k, a_{k+1})$.

(Here we take $a_{n+1} = +\infty$.)

# Persistence modules

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Henceforth, all persistence modules will be indexed by $\mathbb{R}$ unless stated otherwise.

### Tameness

A persistence module is **q-tame** if $v_t^s$ has finite rank whenever $s < t$.

### Fact

q-tame persistence modules have well-behaved persistence diagrams.

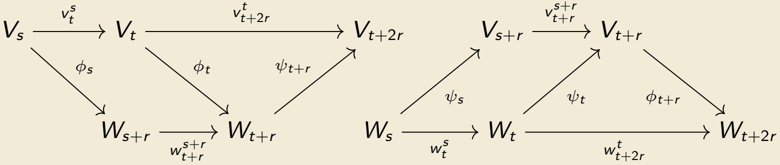
# Persistence modules

## The interleaving distance (Chazal et al. 2009)

An $r$-**interleaving** between persistence modules $V, W$ is a collection of maps

$$\phi_t : V_t \to W_{t+r}, \quad \psi_t : W_t \to V_{t+r}$$

such that the diagrams




commute for all $s < t$.

A 0-interleaving is an isomorphism of persistence modules.

The **interleaving distance** between $V, W$ is

$$d_{\mathrm{I}}(V, W) = \inf_{r \geq 0} \left\{ r \;\middle|\; \text{there exists an } r\text{-interleaving between } V, W \right\}$$

## Simplex streams

Let $S$ be a simplicial complex and let $f, g : S \to \mathbb{R}$ define simplex streams.

Suppose $\|f - g\|_\infty \leq r$. Writing

$$[f]^t = f^{-1}(-\infty, t], \qquad [g]^t = g^{-1}(-\infty, t],$$

we have inclusions

$$[f]^t \subseteq [g]^{t+r}, \qquad [g]^t \subseteq [f]^{t+r}.$$

More generally, we have diagrams

$$\begin{array}{ccccccc} [f]^s & \longrightarrow & [f]^t & & \longrightarrow & & [f]^{t+2r} \\ & \searrow & & \searrow & & \nearrow & \\ & & [g]^{s+r} & \longrightarrow & [g]^{t+r} & & \end{array} \qquad \begin{array}{ccccccc} & & [f]^{s+r} & \longrightarrow & [f]^{t+r} & & \\ & \nearrow & & \nearrow & & \searrow & \\ [g]^s & \longrightarrow & [g]^t & & \longrightarrow & & [g]^{t+2r} \end{array}$$

which automatically commute, since all arrows are inclusion maps.

Applying homology $\mathrm{H}_*$, we obtain the maps $\phi_t, \psi_t$ and required relations for an $r$-interleaving between the persistence modules for the two simplex streams.

## Sublevelset filtrations

Let $M$ be a compact smooth manifold with Morse functions $f, g : M \to \mathbb{R}$.

Suppose $\|f - g\|_\infty \leq r$. Writing

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Applying homology $\mathrm{H}_*$, we obtain the maps $\phi_t, \psi_t$ and required relations for an $r$-interleaving between the persistence modules for the two sublevelset filtrations.

## Sublevelset filtrations

Let $P$ be a compact polyhedron with piecewise linear maps $f, g : P \to \mathbb{R}$.

Suppose $\|f - g\|_\infty \leq r$. Writing

$$[f]^t = f^{-1}(-\infty, t], \qquad [g]^t = g^{-1}(-\infty, t],$$

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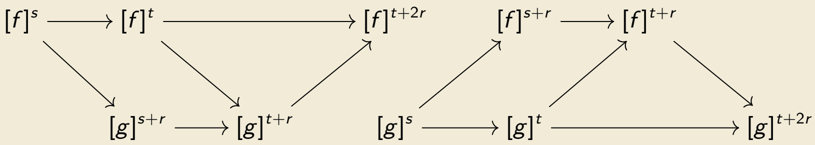



which automatically commute, since all arrows are inclusion maps.

Applying homology $\mathrm{H}_*$, we obtain the maps $\phi_t, \psi_t$ and required relations for an $r$-interleaving between the persistence modules for the two sublevelset filtrations.

# Data $\longrightarrow$ Persistence modules

## Metric data (fixed set)

Let $X$ be a finite set and let $d, e : X \times X \to \mathbb{R}$ be metrics.

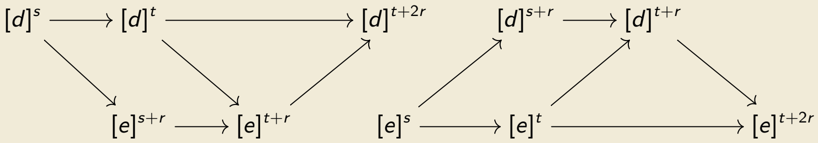
Suppose $\|d - e\|_\infty \leq r$. Writing

$$[d]^t = \mathrm{VR}((X, d), t) \qquad [e]^t = \mathrm{VR}((X, e), t)$$

we have inclusions

$$[d]^t \subseteq [e]^{t+r}, \qquad [e]^t \subseteq [d]^{t+r}.$$

More generally, we have diagrams




which automatically commute, since all arrows are inclusion maps.

Applying homology $\mathrm{H}_*$, we obtain the maps $\phi_t, \psi_t$ and required relations for an $r$-interleaving between the two Vietoris–Rips homology persistence modules.

# Data $\longrightarrow$ Persistence modules

**More subtle**

The other two examples are slightly more subtle:

**Metric data (subsets of a fixed space)**

Let $X, Y \subseteq Z$ be finite subsets of a fixed metric space $(Z, d)$.

**Metric data (general)**

Let $(X, d)$, $(Y, e)$ be finite metric spaces.

# Detour: Multivalued maps

## Multivalued maps

Let $X, Y$ be sets. A **multivalued map** $F : X \rightrightarrows Y$ is a function

$$F : X \to \mathcal{P}(Y)$$

that carries each $x \in X$ to a nonempty subset of $Y$.

## Selections

A **selection** from $F$ is a function $f : X \to Y$ such that $f(x) \in F(x)$ for all $x$.

## Subordinate maps

Let $F, G : X \rightrightarrows Y$. We say that $G$ is **subordinate** to $F$, and write $G \subseteq F$, if

$$G(x) \subseteq F(x)$$

for all $x$. Equivalently, every selection from $G$ is a selection from $F$.

# Detour: Multivalued maps

## Multivalued simplicial maps

Let $S$, $T$ be simplicial complexes with vertex sets $X$, $Y$.
A **multivalued simplicial map** $F : S \rightrightarrows T$ is a multivalued map $F : X \rightrightarrows Y$, such that the image of every simplex $\sigma \in S$

$$F\sigma = \bigcup_{x \in \sigma} F(x)$$

(has the property that every finite subset) is a simplex of $T$.

## Induced homology map

Let $F : S \rightrightarrows T$ be a multivalued simplicial map. Then the maps

$$f_* : \mathrm{H}_*(S) \to \mathrm{H}_*(T)$$

induced by selections $f$ are all equal to each other. Call this map $F_*$.

## Proof

Indeed, any two selections are contiguous.

## Remark

It is immediate that $G \subseteq F$ implies $G_* = F_*$.

**Metric data (subsets of a fixed space)**

Let $X, Y \subseteq Z$ be finite subsets of a fixed metric space $(Z, d)$.

Suppose $d_{\mathrm{H}}(X, Y) \leq r/2$. Then we can define multivalued maps:

$$F : X \rightrightarrows Y; \qquad F(x) = \{y \in Y \mid d(x, y) \leq r/2\}.$$
$$G : Y \rightrightarrows X; \qquad G(y) = \{x \in Y \mid d(y, x) \leq r/2\}.$$

Using the triangle inequality, these define multivalued simplicial maps

$$F : \mathrm{VR}(X, t) \rightrightarrows \mathrm{VR}(Y, t + r),$$
$$G : \mathrm{VR}(Y, t) \rightrightarrows \mathrm{VR}(X, t + r).$$

Applying homology, we get maps

$$\phi_t = F_* : \mathrm{H}_*(\mathrm{VR}(X, t)) \rightrightarrows \mathrm{H}_*(\mathrm{VR}(Y, t + r))$$
$$\psi_t = G_* : \mathrm{H}_*(\mathrm{VR}(Y, t)) \rightrightarrows \mathrm{H}_*(\mathrm{VR}(X, t + r))$$

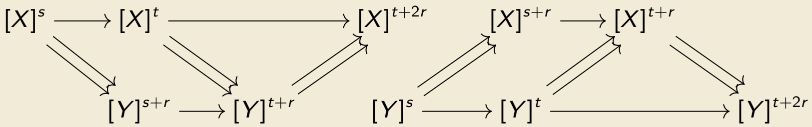
It remains to verify the required relations for an $r$-interleaving.

# Data $\longrightarrow$ Persistence modules

## Metric data (subsets of a fixed space)

Let $X, Y \subseteq Z$ be finite subsets of a fixed metric space $(Z, d)$.

Write $[X]^t = \mathrm{VR}(X, t)$ and $[Y]^t = \mathrm{VR}(Y, t)$. Then we have:




The two parallelograms commute.
The triangles don't commute. However, the horizontal map in each triangle is subordinate to the composite of the two diagonal maps.

Applying homology, the resulting diagrams do commute.
Thus the maps $\phi_t$, $\psi_t$ define an $r$-interleaving.

## Conclusion

subset of a metric space $\longrightarrow$ VR persistent homology is 2-Lipschitz.

# Persistence Diagrams

## Diagrams

A **diagram in** $\mathcal{H}$ is a multiset of points in the region of the extended plane

$$\mathcal{H} = \{(p, q) \mid -\infty < p < q \leq +\infty\}$$

## Point metric

We use the $\ell^\infty$ metric for points in this region:

$$d^\infty((p_1, q_1), (p_2, q_2)) = \begin{cases} \max(|p_1 - p_2|, |q_1 - q_2|) & \text{if } q_1, q_2 \text{ are finite} \\ |p_1 - p_2| & \text{if } q_1 = q_2 = +\infty \\ \infty & \text{otherwise} \end{cases}$$

We also note the distance from the diagonal:

$$d^\infty((p, q), \Delta) = \tfrac{1}{2}(q - p)$$

**The bottleneck distance**

Let $A, B$ be diagrams in $\mathcal{H}$.

An $r$-**matching** is a partial bijection between the multisets $A, B$ such that:

- If $a \in A$ and $b \in B$ are matched, then $d^\infty(a, b) \leq r$.
- If $a \in A$ is not matched, then $d^\infty(a, \Delta) \leq r$.
- If $b \in B$ is not matched, then $d^\infty(b, \Delta) \leq r$.

The **bottleneck distance** between $A, B$ is

$$d_{\mathrm{B}}(A, B) = \inf_{r \geq 0} \left\{ r \;\middle|\; \text{there exists an } r\text{-matching between } A, B \right\}$$

# Persistence modules $\longrightarrow$ Persistence diagrams

**Algebraic Stability Theorem (Chazal et al. 2009)**

If two q-tame persistence modules are $r$-interleaved,
then their persistence diagrams are $r$-matched.

$$\boxed{\text{persistence module}} \longrightarrow \boxed{\text{persistence diagram}} \quad \text{is 1-Lipschitz}$$

**Ingredients**

- Box Lemma.
- Glisse Interpolation Lemma.

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**Box Lemma**

Let $V, W$ be $r$-interleaved persistence modules with persistence diagrams $A, B$.

Then the number of points of $A$ inside any rectangle $R$ is no greater than the number of points of $B$ inside the rectangle $R^r$ obtained by thickening $R$ on both sides by $r$ (provided that $R^r$ does not meet the diagonal).

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**Glisse Interpolation Lemma**

Let $V, W$ be $r$-interleaved persistence modules.

Then there is a 1-parameter family of persistence modules

$$(V_t \mid t \in [0, r])$$

such that $V_0 = V$ and $V_r = W$, with the additional property that $V_s, V_t$ are $|s - t|$-interleaved for all $s, t$.

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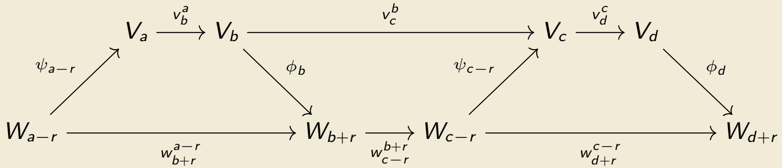
### Outline Proof

Let $R = [a, b] \times [c, d]$ so that $R^r = [a - r, b + r] \times [c - r, d + r]$.

Consider the diagram:



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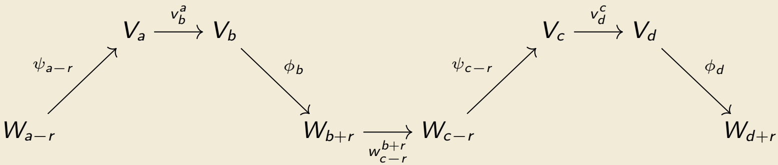

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Let $R = [a, b] \times [c, d]$ so that $R^r = [a - r, b + r] \times [c - r, d + r]$.

$$W_{a-r} \longrightarrow V_a \longrightarrow V_b \longrightarrow W_{b+r} \longrightarrow W_{c-r} \longrightarrow V_c \longrightarrow V_d \longrightarrow W_{d+r}$$

Rectangle point counts are obtained by counting interval summands:

| | | | |
|---|---|---|---|
| | ○●●●●●●○ | ○○●●●●●○ | ○○○●●●●○ |
| $\#\left[B \cap R^r\right]$ : | ○●●●●●○○ | ○○●●●●○○ | ○○○●●●○○ |
| | ○●●●●○○○ | ○○●●●○○○ | ○○○●●○○○ |
| $\#\left[A \cap R\right]$ : | | ○○●●●●○○ | |

**Glisse Interpolation Lemma**

Let $V, W$ be $r$-interleaved persistence modules.

Then there is a 1-parameter family of persistence modules

$$(V_t \mid t \in [0, r])$$

such that $V_0 = V$ and $V_r = W$, with the additional property that $V_s, V_t$ are $|s - t|$-interleaved for all $s, t$.

**Remarks**

- The lemma can be proved by direct algebraic construction; or by applying the Kan Extension Theorem from category theory.
- The lemma can be circumvented if the initial data can be interpolated.

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## Simplex streams

Let $S$ be a simplicial complex and let $f, g : S \to \mathbb{R}$ define simplex streams.
Set $f_t = (1 - t)f + tg$.

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### Sublevelset filtrations

Let $M$ be a compact smooth manifold with Morse functions $f, g : M \to \mathbb{R}$.
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Let $P$ be a compact polyhedron with piecewise linear maps $f, g : P \to \mathbb{R}$.
Set $f_t = (1 - t)f + tg$.

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### Metric data (fixed set)

Let $X$ be a finite set and let $d, e : X \times X \to \mathbb{R}$ be metrics.
Set $d_t = (1 - t)d + te$.

### Glisse Interpolation Lemma

Let $V, W$ be $r$-interleaved persistence modules.

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### Metric data (subsets of a fixed space)

Let $X, Y \subseteq Z$ be finite subsets of a fixed metric space $(Z, d)$.
Set

$$X_t = \{(1 - t)x + ty \mid x \in X,\ y \in Y,\ d(x, y) \leq d_{\mathrm{H}}(X, Y)\}.$$

## Outline proof of Algebraic Stability Theorem

- The Box Lemma can be used to show that

  $$\boxed{\text{persistence module}} \longrightarrow \boxed{\text{persistence diagram}}$$

  is locally 1-Lipschitz (near persistence modules with a finite diagram).
- Local 1-Lipschitz implies global 1-Lipschitz, by applying a Heine–Borel argument along a path provided by the Glisse Interpolation Lemma.