5. Persistence Diagrams

Vin de Silva
Pomona College

CBMS Lecture Series, Macalester College, June 2017
Persistence modules

A diagram of vector spaces and linear maps

\[ V_0 \to V_1 \to \ldots \to V_n \]

is called a **persistence module**.
Persistence modules

A diagram of vector spaces and linear maps

\[ V_0 \rightarrow V_1 \rightarrow \ldots \rightarrow V_n \]

is called a **persistence module**.

**Example**

Let

\[ S_0 \subseteq S_1 \subseteq \ldots \subseteq S_n \]

be a nested sequence of simplicial complexes. Then

\[ H_k(S_0) \rightarrow H_k(S_1) \rightarrow \ldots \rightarrow H_k(S_n) \]

is a persistence module. The arrows denote the maps in homology induced by the inclusions \( S_{k-1} \subseteq S_k \).
Persistence modules

A diagram of vector spaces and linear maps

\[ V_0 \rightarrow V_1 \rightarrow \ldots \rightarrow V_n \]

is called a **persistence module**.

Example

Let \( X \) be a finite metric space and let

\[ r_0 < r_1 < \cdots < r_n \]

be the set of values taken by the metric. Then

\[ H_k(\text{VR}(X, r_0)) \rightarrow H_k(\text{VR}(X, r_1)) \rightarrow \ldots \rightarrow H_k(\text{VR}(X, r_n)) \]

is a persistence module. The arrows denote the maps in homology induced by the inclusions \( \text{VR}(X, r_{k-1}) \subseteq \text{VR}(X, r_k) \).
A diagram of vector spaces and linear maps

\[ V_0 \rightarrow V_1 \rightarrow \ldots \rightarrow V_n \]

is called a **persistence module**.

**Example**

Let \( M \) be a compact smooth manifold and let \( f : M \rightarrow \mathbb{R} \) be a smooth function with finitely many critical values \( t_0 < t_1 < \ldots < t_n \).

Let \( M^t = f^{-1}(-\infty, t] \). Then

\[ H_k(M^{t_0}) \rightarrow H_k(M^{t_1}) \rightarrow \ldots \rightarrow H_k(M^{t_n}) \]

is a persistence module. The arrows denote the maps in homology induced by the inclusions \( M^{t_{k-1}} \subseteq M^{t_k} \).
A persistence module indexed by $\mathbb{N}$ is an infinite diagram

$$V_0 \rightarrow V_1 \rightarrow \ldots \rightarrow V_n \rightarrow \ldots$$

of vector spaces and linear maps.

Zomorodian, Carlsson (2002)

This is the same thing as a graded module over the polynomial ring $\mathbb{F}[t]$.

Field elements act on each $V_k$ by scalar multiplication. The ring element $t$ acts by applying the map $V_k \rightarrow V_{k+1}$.
Diagrams of the form $V_0$

are classified up to isomorphism by a single numerical invariant:

$$\dim(V_0)$$
The rank invariant

Diagrams of the form $V_0$

are classified up to isomorphism by a single numerical invariant:

$$r(0, 0) = \text{rank}(V_0 \rightarrow V_0) = \text{dim}(V_0)$$
The rank invariant

Diagrams of the form

\[ V_0 \rightarrow V_1 \]

are classified up to isomorphism by three numerical invariants:

\[ \dim(V_0) \]
\[ \operatorname{rank}(V_0 \rightarrow V_1) \]
\[ \dim(V_1) \]

Proof

Represent the map by a matrix. Changes of basis of \( V_0, V_1 \) correspond to column and row operations. Every matrix can be reduced to the block form:

\[
\begin{bmatrix}
I_r & 0 \\
0 & 0
\end{bmatrix}
\]

These block forms are precisely classified by the three invariants.
The rank invariant

Diagrams of the form

\[ V_0 \rightarrow V_1 \]

are classified up to isomorphism by three numerical invariants:

\[
\begin{align*}
  r(0, 0) &= \dim(V_0) = \rank(V_0 \rightarrow V_0) \\
  r(0, 1) &= \rank(V_0 \rightarrow V_1) \\
  r(1, 1) &= \dim(V_1) = \rank(V_1 \rightarrow V_1)
\end{align*}
\]

Proof

Represent the map by a matrix. Changes of basis of \( V_0, V_1 \) correspond to column and row operations. Every matrix can be reduced to the block form:

\[
\begin{bmatrix}
  I_r & 0 \\
  0 & 0
\end{bmatrix}
\]

These block forms are precisely classified by the three invariants.
The rank invariant

Theorem (Zomorodian, Carlsson, 2002)

Diagrams of the form

\[ V_0 \rightarrow V_1 \rightarrow \ldots \rightarrow V_n \]

are classified up to isomorphism by the invariants

\[ r(i, j) = \text{rank}(V_i \rightarrow V_j) \]

for all \( 0 \leq i \leq j \leq n \).
The rank invariant

Theorem (Zomorodian, Carlsson, 2002)

Diagrams of the form

\[ V_0 \rightarrow V_1 \rightarrow \ldots \rightarrow V_n \]

are classified up to isomorphism by the invariants

\[ r(i, j) = \text{rank}(V_i \rightarrow V_j) \]

for all \( 0 \leq i \leq j \leq n \).

Plot these as points-with-multiplicity on a ‘rank diagram’:

\[ \begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
\bullet & \bullet & \bullet & \bullet & \bullet & 4 \\
\bullet & \bullet & \bullet & 3 \\
\bullet & \bullet & 2 \\
\bullet & 1 \\
\bullet & 0 \\
\end{array} \]
Interval decomposition

Direct sums

Given two persistence modules

\[ V_0 \rightarrow V_1 \rightarrow \ldots \rightarrow V_n \]

\[ W_0 \rightarrow W_1 \rightarrow \ldots \rightarrow W_n \]

we can form their direct sum:

\[ V_0 \oplus W_0 \rightarrow V_1 \oplus W_1 \rightarrow \ldots \rightarrow V_n \oplus W_n \]

Direct sum decomposition

Given a persistence module

\[ V_0 \rightarrow V_1 \rightarrow \ldots \rightarrow V_n \]

can it be expressed as the direct sum of two non-trivial persistence modules? If it can’t, then it is indecomposable.
Interval decomposition

Interval modules

The **interval module** $I(i, j)$ is defined to be the persistence module with

$$V_k = \begin{cases} F & \text{if } i \leq k \leq j \\ 0 & \text{otherwise} \end{cases}$$

and all maps $F \to F$ being the identity (the other maps necessarily being zero).

Here are examples of **interval modules** with $n = 5$:

$$
\begin{array}{c}
0 \to 0 \to F \overset{1}{\to} F \overset{1}{\to} F \to 0 \\
0 \to 0 \to F \overset{1}{\to} F \overset{1}{\to} F \to F \\
F \overset{1}{\to} F \overset{1}{\to} F \overset{1}{\to} F \overset{1}{\to} F \to F
\end{array}
$$
Interval decomposition

Interval modules

The **interval module** $I(i,j)$ is defined to be the persistence module with

$$V_k = \begin{cases} F & \text{if } i \leq k \leq j \\ 0 & \text{otherwise} \end{cases}$$

and all maps $F \to F$ being the identity (the other maps necessarily being zero).

Here are examples of **interval modules** with $n = 5$:

0 → 0 → $F$ → $F$ → $F$ → 0

0 → 0 → $F$ → $F$ → $F$ → $F$

$F$ → $F$ → $F$ → $F$ → $F$ → $F$

**Fact**

Interval modules are indecomposable.
Interval decomposition

Theorem (Gabriel 1970; Zomorodian & Carlsson 2002; etc.)

Every (finite-dimensional) persistence module is isomorphic to a (finite) direct sum of interval modules. The multiplicity

\[ m(i, j) \]

of the summand \( I(i, j) \) is an invariant of the persistence module.
## Theorem (Gabriel 1970; Zomorodian & Carlsson 2002; etc.)

Every **(finite-dimensional)** persistence module is isomorphic to a **(finite)** direct sum of interval modules. The multiplicity

\[ m(i, j) \]

of the summand \( I(i, j) \) is an invariant of the persistence module.

## Proof

Zomorodian & Carlsson use the classification theorem for finitely-generated graded modules over \( \mathbb{F}[t] \).
Theorem (Gabriel 1970; Zomorodian & Carlsson 2002; etc.)

Every (finite-dimensional) persistence module is isomorphic to a (finite) direct sum of interval modules. The multiplicity

\[ m(i, j) \]

of the summand \( I(i, j) \) is an invariant of the persistence module.

Proof

Zomorodian & Carlsson use the classification theorem for finitely-generated graded modules over \( \mathbb{F}[t] \).

Plot the multiplicities as points-with-multiplicity on a ‘persistence diagram’:
Relationship between the two diagrams

Inverse transformations

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r(i, j))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ r(i, j) = \sum_{p=0}^{i} \sum_{q=i}^{j} m(p, q) \]

The inverse transformation is

\[ m(i, j) = r(i, j) - r(i-1, j) - r(i, j+1) + r(i-1, j+1) \]

by the inclusion exclusion principle.
Relationship between the two diagrams

Inverse transformations

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r(i,j) )</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

\[ \begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array} \]

\[ \begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array} \]

\[ \begin{array}{cccccc}
4 & 3 & 2 & 1 & 0 \\
4 & 3 & 2 & 1 & 0 \\
4 & 3 & 2 & 1 & 0 \\
4 & 3 & 2 & 1 & 0 \\
4 & 3 & 2 & 1 & 0 \\
\end{array} \]

\[ \begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 2 & 3 & 4 \\
\end{array} \]

rank diagram \( \leftrightarrow \) persistence diagram

Each summand \( I(p, q) \) contributes to \( r(i,j) \) whenever \( p \leq i \leq j \leq q \). Thus:

\[
r(i,j) = \sum_{p=0}^{i} \sum_{q=j}^{n} m(p, q)
\]
Relationship between the two diagrams

Inverse transformations

<table>
<thead>
<tr>
<th>0 1 2 3 4</th>
<th>r(i, j)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>0 1 2 3 4</th>
<th>m(i, j)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

rank diagram ← persistence diagram

Each summand \( I(p, q) \) contributes to \( r(i, j) \) whenever \( p \leq i \leq j \leq q \). Thus:

\[
r(i, j) = \sum_{p=0}^{i} \sum_{q=j}^{n} m(p, q)
\]

rank diagram → persistence diagram

The inverse transformation is

\[
m(i, j) = r(i, j) - r(i - 1, j) - r(i, j + 1) + r(i - 1, j + 1)
\]

by the inclusion exclusion principle.
### Inverse transformations

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{array}{c|c|c|c|c|c}
0 & 1 & 2 & 3 & 4 & r(i, j) \\
\hline
0 & 1 & 2 & 3 & 4 & m(i, j)
\end{array}
\]

### Rank diagram $\leftarrow$ Persistence diagram

Each summand \( I(p, q) \) contributes to \( r(i, j) \) whenever \( p \leq i \leq j \leq q \). Thus:

\[
r(i, j) = \sum_{p=0}^{i} \sum_{q=j}^{n} m(p, q)
\]

### Rank diagram $\rightarrow$ Persistence diagram

The inverse transformation is

\[
m(i, j) = r(i, j) - r(i - 1, j) - r(i, j + 1) + r(i - 1, j + 1)
\]

by the inclusion-exclusion principle.
Plotting the persistence diagram

Plotting conventions

For persistence modules such as the Vietoris–Rips homology of a finite metric space, constructed with respect to a set of critical values

\[ r_0 < r_1 < \cdots < r_n \]

we adopt the following convention. Each summand \( I(i,j) \) is drawn as

- a half-open interval \([r_i, r_{j+1})\) in the barcode; or
- a point \((r_i, r_{j+1})\) in the persistence diagram.

We set \( r_{n+1} = +\infty \) by convention.
Given a nested sequence of simplicial complexes

\[ S_0 \subseteq S_1 \subseteq \ldots \subseteq S_n \]

we wish to compute the persistence diagram of

\[ H_d(S_0) \to H_d(S_1) \to \ldots \to H_d(S_n). \]

We compute it for all \( d \) simultaneously (up to some maximum dimension).
The ELZ Persistence Algorithm

Edelsbrunner, Letscher, Zomorodian (2000)

Given a nested sequence of simplicial complexes

\[ S_0 \subseteq \rightarrow S_1 \subseteq \rightarrow \cdots \subseteq \rightarrow S_n \]

we wish to compute the persistence diagram of

\[ \text{H}_d(S_0) \rightarrow \text{H}_d(S_1) \rightarrow \cdots \rightarrow \text{H}_d(S_n). \]

We compute it for all \( d \) simultaneously (up to some maximum dimension).

**Input**

A sequence of ‘cells’ \( \sigma_0, \sigma_1, \ldots, \sigma_n \). For each cell we are given:

- the dimension \( d = \text{dim}(\sigma_k) \);
- the boundary \( \partial[\sigma_k] \), as a cycle built from existing \((d - 1)\)-cells.

**Output**

The persistence diagram for this nested sequence of formal cell-complexes.
The ELZ Persistence Algorithm

Data structure

We maintain an upper-triangular square matrix, size equal to the number of cells processed so far. Columns are coloured green, yellow, red.

<table>
<thead>
<tr>
<th>colour</th>
<th>content</th>
<th>leading coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>green</td>
<td>$d$-cycle $\alpha_k$</td>
<td>1</td>
</tr>
<tr>
<td>yellow</td>
<td>$d$-cycle $\beta_k$</td>
<td>1</td>
</tr>
<tr>
<td>red</td>
<td>$d$-chain $\gamma_k$</td>
<td>non-zero</td>
</tr>
</tbody>
</table>

Here $d$ is the dimension of the cell $\sigma_k$ whose column it is.

We maintain a bijection between the yellow and red columns. If yellow column $k$ is paired with red column $\ell$, then $k < \ell$ and $\partial \gamma_{\ell} = \beta_k$.

Readout

Each green column $j$ corresponds to an interval $[j, +\infty)$. Each yellow–red pair $(k, \ell)$ corresponds to an interval $[k, \ell)$.
The ELZ Persistence Algorithm

**Matrix**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>12</th>
<th>23</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>*</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Chains**

- **green**: \( \alpha_0 = [1], \alpha_2 = [3] \)
- **yellow**: \( \beta_1 = [2] - [1] \)
- **red**: \( \gamma_3 = [1, 2] \)

**Persistence intervals**

\([0, +\infty), [1, 3), [2, +\infty)\)
The ELZ Persistence Algorithm

**Lemma**

The **green** and **yellow** columns form a basis for the space of all cycles. The **yellow** columns form a basis for the space of all boundaries.
The ELZ Persistence Algorithm

Lemma
The green and yellow columns form a basis for the space of all cycles. The yellow columns form a basis for the space of all boundaries.

Cycle reduction
Using the table, any cycle $\psi$ can be uniquely written

$$\psi = \partial \rho \quad \text{or} \quad \psi = \partial \rho + \phi$$

$\rho$ is a combination of red columns, $\phi$ is a cycle whose leading simplex is green.
# The ELZ Persistence Algorithm

## Lemma

The **green** and **yellow** columns form a basis for the space of all cycles. The **yellow** columns form a basis for the space of all boundaries.

## Cycle reduction

Using the table, any cycle $\psi$ can be uniquely written

\[ \psi = \partial \rho \quad \text{or} \quad \psi = \partial \rho + \phi \]

$\rho$ is a combination of **red** columns, $\phi$ is a cycle whose leading simplex is **green**.

## Iterative method

Initialise $\rho = 0$. Consider $\psi - \partial \rho$.

- If it is zero then STOP.
- If its leading simplex is **green**, then set $\phi = \psi - \partial \rho$ and STOP.
- If its leading simplex is **yellow**, then add to $\rho$ the appropriate multiple of the associated **red** column to eliminate that leading term.

Repeat.
Update Step

Consider the next simplex $\sigma_k$. 

If $\partial[\sigma_k]$ was already a boundary:
The cycle reduction lemma gives $\partial[\sigma_k] = \partial\rho$. 

We get a new green cycle $\alpha_k = [\sigma_k] - \rho$.

If $\partial[\sigma_k]$ was not previously a boundary:
The cycle reduction lemma gives $\partial[\sigma_k] = \partial\rho + \phi$ where the leading term of $\phi$ is $c[\sigma_j]$ for some scalar $c$.

Change the $j$-th column to yellow, and colour the $k$-th column red. Set $\beta_j = c - 1 \phi$, $\gamma_k = c - 1 ([\sigma_k] - \rho)$ (discarding $\alpha_j$). Pair the columns $j$ and $k$. 

Vin de Silva Pomona College
The **ELZ Persistence Algorithm**

**Update Step**

Consider the next simplex $\sigma_k$.

*If $\partial[\sigma_k]$ was already a boundary:* The cycle reduction lemma gives

$$\partial[\sigma_k] = \partial \rho.$$  

We get a new **green** cycle

$$\alpha_k = [\sigma_k] - \rho.$$
The ELZ Persistence Algorithm

Update Step

Consider the next simplex $\sigma_k$.

If $\partial[\sigma_k]$ was already a boundary: The cycle reduction lemma gives

$$\partial[\sigma_k] = \partial\rho.$$ 

We get a new green cycle

$$\alpha_k = [\sigma_k] - \rho.$$ 

If $\partial[\sigma_k]$ was not previously a boundary: The cycle reduction lemma gives

$$\partial[\sigma_k] = \partial\rho + \phi$$

where the leading term of $\phi$ is $c[\sigma_j]$ for some scalar $c$. Change the $j$-th column to yellow, and colour the $k$-th column red. Set

$$\beta_j = c^{-1}\phi, \quad \gamma_k = c^{-1}([\sigma_k] - \rho)$$

(discarding $\alpha_j$). Pair the columns $j$ and $k$. 
### The ELZ Persistence Algorithm

#### Matrix

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>12</th>
<th>23</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Chains

- **green**: $\alpha_0 = [1], \alpha_2 = [3]$
- **yellow**: $\beta_1 = [2] - [1]$
- **red**: $\gamma_3 = [1, 2]$

#### Persistence intervals

$[0, +\infty), \ [1, 3), \ [2, +\infty)$
The ELZ Persistence Algorithm

Matrix

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>12</th>
<th>23</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td>*</td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td></td>
<td></td>
<td>*</td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Chains

green : \( \alpha_0 = [1] \)


green : \( \gamma_3 = [1, 2], \gamma_4 = [2, 3] \)

Persistence intervals

\([0, +\infty), [1, 3), [2, 4)\)
The ELZ Persistence Algorithm

Matrix

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>12</th>
<th>23</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td>*</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>23</td>
<td></td>
<td></td>
<td></td>
<td>*</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

Chains

**green**: \( \alpha_0 = [1], \alpha_5 = [1, 3] − [2, 3] − [1, 2] \)
**yellow**: \( \beta_1 = [2] − [1], \beta_2 = [3] − [2] \)
**red**: \( \gamma_3 = [1, 2], \gamma_4 = [2, 3] \)

Persistence intervals

\([0, +\infty), \ [1, 3), \ [2, 4), \ [5, +\infty)\)