3. Persistence

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The homology of a planar region

Chain complex

We have constructed a diagram of vector spaces and linear maps

$$0 \xleftarrow{\partial_{-1}} \mathrm{C}_0(U) \xleftarrow{\partial_0} \mathrm{C}_1(U) \xleftarrow{\partial_1} \mathrm{C}_2(U)$$

for any plane region $U \subseteq \mathbb{R}^2$.

From this we define

$$\begin{aligned} \mathrm{Z}_k(U) &= \ker(\partial_{k-1}) && \text{the space of } k\text{-cycles} \\ \mathrm{B}_k(U) &= \mathrm{im}(\partial_k) && \text{the space of } k\text{-boundaries} \end{aligned}$$

Since $\partial_{k-1}\partial_k = 0$, we have $\mathrm{B}_k \leq \mathrm{Z}_k$. Since $\partial_{-1} = 0$ we have $\mathrm{Z}_0 = \mathrm{C}_0$.

Homology = Cycles/Boundaries

$$\mathrm{H}_0(U) = \frac{\ker(\partial_{-1})}{\mathrm{im}(\partial_0)} = \frac{\mathrm{Z}_0(U)}{\mathrm{B}_0(U)} = \frac{\mathrm{C}_0(U)}{\mathrm{B}_0(U)}$$

$$\mathrm{H}_1(U) = \frac{\ker(\partial_0)}{\mathrm{im}(\partial_1)} = \frac{\mathrm{Z}_1(U)}{\mathrm{B}_1(U)}$$

The homology of a planar region

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$$\mathrm{H}_1(U) = \frac{\ker(\partial_0)}{\mathrm{im}(\partial_1)} = \frac{\mathrm{Z}_1(U)}{\mathrm{B}_1(U)}$$

The **Betti numbers** of U are defined as follows:

$$\mathrm{b}_0(U) = \dim(\mathrm{H}_0(U))$$

$$\mathrm{b}_1(U) = \dim(\mathrm{H}_1(U))$$

The homology of a planar region

Theorem

Let $U \subseteq \mathbb{R}^2$ be open. Then $\mathrm{b}_0(U) = \#$ connected components of U.

The homology of a planar region

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Proof

For simplicity assume finitely many components $U_1, \ldots, U_k$.
Select points $p_i \in U_i$. Then $[p_i] \in \mathrm{C}_0(U)$.

- $[p_1], \ldots, [p_k]$ span $\mathrm{C}_0(U)$ modulo $\mathrm{B}_0(U)$.
- $[p_1], \ldots, [p_k]$ are linearly independent modulo $\mathrm{B}_0(U)$.

We must show that every 0-chain can be written in the form

$$\alpha = \lambda_1[p_1] + \cdots + \lambda_k[p_k] + \partial\gamma.$$

Indeed, every generator can be written in this form. Specifically,

$$[a] = [p_i] + \partial\gamma$$

where

$$\gamma = [p_i, a_1] + [a_1, a_2] + \cdots + [a_n, a]$$

for some polygonal path $P(p_i, a_1, a_2, \ldots, a_n, a)$ in U.

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Suppose

$$\lambda_1[p_1] + \cdots + \lambda_k[p_k] + \partial\gamma = 0$$

Consider the linear maps $\mu_i : \mathrm{C}_0(U) \to \mathbb{F}$ defined on generators by:

$$\mu_i([a]) = \begin{cases} 1 & \text{if } a \in U_i \\ 0 & \text{if } a \notin U_i \end{cases}$$

Then $\mu_i \partial = 0$ since the endpoints of each edge must lie in the same component.
Applying μ_i to the equation yields $\lambda_i = 0$.

The homology of a planar region

Theorem

Let $U = \mathbb{R}^2 - \{a_1, a_2, \ldots, a_\ell\}$ where the a_i are distinct. Then $\mathrm{b}_1(U) = \ell$.

The homology of a planar region

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Proof

Let $\gamma_i \in \mathrm{Z}_1(U)$ denote a small counterclockwise square around a_i.

- $\gamma_1, \ldots, \gamma_\ell$ span $\mathrm{Z}_1(U)$ modulo $\mathrm{B}_1(U)$.
- $\gamma_1, \ldots, \gamma_\ell$ are linearly independent modulo $\mathrm{B}_1(U)$.

Suppose

$$\lambda_1\gamma_1 + \cdots + \lambda_\ell\gamma_\ell + \partial\sigma = 0$$

Consider the linear maps $\mathrm{w}_i : \mathrm{Z}_1(U) \to \mathbb{F}$ defined by:

$$\mathrm{w}_i(\zeta) = \mathrm{w}(\zeta, a_i).$$

We have seen that $\mathrm{w}_i\,\partial = 0$ in $\mathbb{R}^2 - \{a_1, \ldots, a_\ell\}$.
Applying w_i to the equation yields $\lambda_i = 0$.

Theorem

Let $U = \mathbb{R}^2 - \{a_1, a_2, \ldots, a_\ell\}$ where the a_i are distinct. Then $\mathrm{b}_1(U) = \ell$.

Proof

Let $\gamma_i \in \mathrm{Z}_1(U)$ denote a small counterclockwise square around a_i.

- $\gamma_1, \ldots, \gamma_\ell$ span $\mathrm{Z}_1(U)$ modulo $\mathrm{B}_1(U)$.
- $\gamma_1, \ldots, \gamma_\ell$ are linearly independent modulo $\mathrm{B}_1(U)$.

We must show that every 1-cycle can be written in the form

$$\zeta = \lambda_1\gamma_1 + \cdots + \lambda_\ell\gamma_\ell + \partial\sigma.$$

Write $\lambda_i = \mathrm{w}(\zeta, a_i)$ and replace ζ by $\zeta - (\lambda_1\gamma_1 + \cdots + \lambda_\ell\gamma_\ell)$. This has winding number zero about every a_i.

Partition the plane into a fine rectangular grid. By subtracting boundaries, we can reduce ζ to a cycle of grid edges. By subtracting boundaries of grid rectangles, we can reduce to the case that $\mathrm{w}(\zeta, a) = 0$ for every non-grid point a. By considering rays, it follows that we are reduced to the case $\zeta = 0$.

The homology of a planar region

Dual bases

We have dual bases in the two proofs.

We have

$$[p_1], [p_2], \ldots, [p_k] \quad \text{and} \quad \mu_1, \mu_2, \ldots, \mu_k$$

such that

$$\mu_i \partial = 0 \quad \text{and} \quad \mu_i[p_j] = \begin{cases} 1 & \text{if } i = j \\ 0 & i \neq j \end{cases}$$

which guarantees that $\mathrm{b}_0 \geq k$.

We have

$$\gamma_1, \gamma_2, \ldots, \gamma_\ell \quad \text{and} \quad \mathsf{w}_1, \mathsf{w}_2, \ldots, \mathsf{w}_\ell$$

such that

$$\mathsf{w}_i \, \partial = 0 \quad \text{and} \quad \mathsf{w}_i \, \gamma_j = \begin{cases} 1 & \text{if } i = j \\ 0 & i \neq j \end{cases}$$

which guarantees that $\mathrm{b}_1 \geq \ell$.

For the upper bounds, we used ad hoc geometric constructions.

The cohomology of a planar region

Cochains

A k-**cochain** is a linear map $T : \mathrm{C}_k \to \mathbb{F}$. The collection of all k-cochains is denoted by the symbol C^k:

$$\mathrm{C}^k(U;\mathbb{F}) = \mathrm{Hom}(\mathrm{C}_k(U;\mathbb{F}),\mathbb{F})$$

Cocycles

A k-cochain T is a k-**cocycle** if $T\partial = 0$. A k-cocycle can be thought of as a linear map

$$T : (\mathrm{C}_k/\mathrm{B}_k) \to \mathbb{F}.$$

This restricts to a linear map

$$T : (\mathrm{Z}_k/\mathrm{B}_k) \to \mathbb{F}.$$

The collection of k-cocycles is denoted Z^k.

Coboundaries

A k-cochain of the form $T = S\partial$ is called a k-**coboundary**. Such a cochain restricts to the zero map on Z_k and hence on $\mathrm{H}_k = \mathrm{Z}_k/\mathrm{B}_k$.
The collection of k-coboundaries is denoted B^k. We have $\mathrm{B}^k \subseteq \mathrm{Z}^k \subseteq \mathrm{C}^k$.

The cohomology of a planar region

Cohomology

We define the k-cohomology to be $\mathrm{H}^k = \mathrm{Z}^k/\mathrm{B}^k$.

Duality Theorem

The vector space H^k is the dual of H_k:

$$\mathrm{H}^k \cong \mathrm{Hom}(\mathrm{H}_k, \mathbb{F})$$

Construction of the isomorphism

We have seen that a cocycle defines a map on homology. We get a linear map

$$\mathrm{Z}^k \to \mathrm{Hom}(\mathrm{Z}_k/\mathrm{B}_k, \mathbb{F}).$$

Since coboundaries define the zero map, we get

$$(\mathrm{Z}^k/\mathrm{B}^k) \to \mathrm{Hom}(\mathrm{Z}_k/\mathrm{B}_k, \mathbb{F}).$$

It is not difficult to show that the map is surjective and injective.

Main Lemma: A linear map defined on a subspace can be extended to the whole vector space. (Fails for modules over a commutative ring.)

Example: winding number cocycles

Define cochains $W, X_\phi : \mathrm{C}_1(\mathbb{R}^2 - \{0\}) \to \mathbb{R}$ by

$$W[a,b] = \tfrac{1}{2\pi}\Lambda([a,b],0)$$

$$X_\phi[a,b] = \begin{cases} +1 & \text{if } [a,b] \text{ crosses } R_\phi \text{ counterclockwise} \\ -1 & \text{if } [a,b] \text{ crosses } R_\phi \text{ clockwise} \\ 0 & \text{if } [a,b] \text{ doesn't meet } R_\phi \end{cases}$$

They are cocycles, so they define maps $\mathrm{H}_1 \to \mathbb{R}$.

They differ by a coboundary

$$W = (\tfrac{1}{2\pi}\overline{\arg}_\phi)\partial + X_\phi$$

thanks to the formula

$$\Lambda([a,b],0) = \overline{\arg}_\phi(b) - \overline{\arg}_\phi(a) + 2\pi\xi_j$$

so they define the same map on H_1.

Homology and cohomology of a planar region

Dual complexes

We have defined a homology chain complex

$$0 \longleftarrow \mathrm{C}_0(U) \xleftarrow{\partial_0} \mathrm{C}_1(U) \xleftarrow{\partial_1} \mathrm{C}_2(U)$$

and its dual, a cohomology cochain complex

$$0 \longrightarrow \mathrm{C}^0(U) \xrightarrow{\delta^0} \mathrm{C}^1(U) \xrightarrow{\delta^1} \mathrm{C}^2(U)$$

using algebraic versions of points, edges, and triangles.

(Here δ^k is the operation 'pre-compose with ∂_k'.)

Our homology theory is the homology of 'linear singular simplices'.

Abstract simplicial complexes

A simplicial complex is a set S of nonempty finite sets which is closed under taking subsets:

$$\sigma \in S \text{ and } \emptyset \neq \tau \subseteq \sigma \text{ implies } \tau \in S$$

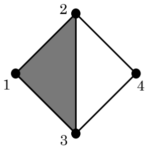
Example

The set

$$M = \{\{2,4\}, \{3,4\}, \{1,2,3\}\}$$

is not a simplicial complex, but its closure under taking subsets is:

$$S = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}\}$$



We write $S = \langle 1, 2, 3, 4, 12, 13, 24, 34, 123 \rangle = \langle 24, 34, 123 \rangle$.

Simplicial complexes

Vertices

The **vertex set** of a simplicial complex S is

$$V = V(S) = \bigcup S$$

In the example above, $V = \{1, 2, 3, 4\}$.

Simplices

An element of S is called a **simplex** (plural **simplices**).
If $\sigma \in S$ has $k + 1$ elements, we call it a k-simplex, and write $\dim \sigma = k$.

Subcomplexes

A **subcomplex** is a subset of a simplical complex that is a simplicial complex.
The k-**skeleton** of S is the subcomplex of simplices of dimension at most k.

Simplicial complexes

Geometric realisation

Let S be a simplicial complex with vertex set V. Its **geometric realization** is the subspace of $\mathbb{R}^V$ defined as follows:

$$|S| = \left\{ f : V \to \mathbb{R} \;\middle|\; f \geq 0,\ f^{-1}(\mathbb{R} - \{0\}) \in S,\ \textstyle\sum_{v \in V} f(v) = 1 \right\}$$

Each vertex w corresponds to a delta function: $\delta_w(v) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{if } v \neq w \end{cases}$

(Caution: If V is infinite, there are two reasonable topologies on $|S|$: the product topology and the Whitehead topology. They agree when S is locally finite.)

Shadow map

Any function $\phi : V \to \mathbb{R}^n$ gives rise to a map $\Phi : |S| \to \mathbb{R}^n$, defined by the formula

$$\Phi(f) = \sum_{v \in V} f(v)\phi(v).$$

It is a continuous map (with respect to the Whitehead topology).

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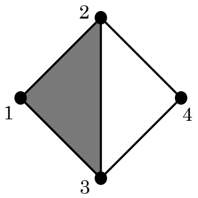


Philosophy

In topological data analysis, we work with abstract simplicial complexes.
If the vertices are taken from data points in $\mathbb{R}^n$, then we may also consider the shadow map. This is rarely injective.

Simplicial complexes

Vietoris–Rips complex

Let X be a metric space, and let $R \geq 0$. The **Vietoris–Rips complex** with diameter R is the simplicial complex with vertex set X defined by the condition:

$$\sigma = \{x_0, x_1, \ldots, x_n\} \in \mathrm{VR}(X, R) \;\Leftrightarrow\; d(x_i, x_j) \leq R \text{ for all } 0 \leq i, j \leq n$$

It is the **clique complex** for the neighbourhood graph of diameter R.

Example

Let X be a finite collection of robotic sensors located in the plane.
The Coverage Theorem uses the 2-skeleton:

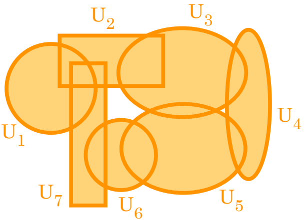
$$\mathrm{VR}(X, R)^{(2)} \subseteq \mathrm{VR}(X, R)$$

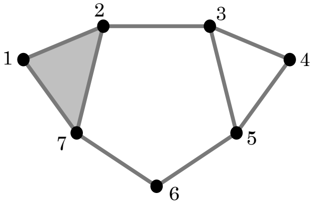
The fence cycle γ is built from 1-simplices.
Coverage is determined in terms of (the shadows of) the 2-simplices.

The nerve of a set system

Let $U_1, \ldots, U_n$ be sets, and write $\mathcal{U} = (U_1, \ldots, U_n)$. The **nerve** of $\mathcal{U}$ is a simplicial complex on the vertex set $\{1, 2, \ldots, n\}$ defined by the following condition:

$$\sigma \in \mathrm{Nerve}(\mathcal{U}) \ \Leftrightarrow \ \bigcap_{k \in \sigma} U_k \neq \emptyset$$





The Nerve Theorem (Leray, Borsuk, etc)

For a family of convex subsets of a vector space, the nerve is a good model for the union of the family.

Simplicial complexes

Čech complex

Let $X \subset \mathbb{R}^N$, and let $r \geq 0$. The **Čech complex** with radius r is the simplicial complex with vertex set X defined by the condition:

$$\sigma = \{x_0, x_1, \ldots, x_n\} \in \text{Čech}(X, r)$$

$$\Leftrightarrow \text{ there exists } p \in \mathbb{R}^N \text{ with } |p - x_i| \leq r \text{ for all } i$$

It is the nerve of the set of closed disks with centers x_i and radius r.
By the Nerve theorem, it is a good model for the union of those disks.

Dowker complex

Let X, Y be sets, let $\phi : X \times Y \to \mathbb{R}$, and let $t \in \mathbb{R}$. The **Dowker complex** with parameter t is the simplicial complex with vertex set X defined by the condition:

$$\sigma = \{x_0, x_1, \ldots, x_n\} \in \text{Dowker}(\phi, t)$$

$$\Leftrightarrow \text{ there exists } y \in Y \text{ with } \phi(x_i, y) \leq t \text{ for all } i$$

Dowker duality

$\text{Dowker}(\phi, t)$ and $\text{Dowker}(\phi^{\mathrm{T}}, t)$ are good models for each other.

Comparing the Vietoris–Rips and Čech complexes

Let $X \subset \mathbb{R}^N$. Then

$$\text{Čech}(X, r) \subseteq \text{VR}(X, 2r)$$

and

$$\text{VR}(X, R) \subseteq \text{Čech}(X, R\sqrt{N/(2N+2)})$$

The second inclusion is Jung's Theorem.

Philosophy

In this way the two methods of simplicial approximation are **interleaved**.

In Euclidean space of any dimension, we deduce

$$\text{VR}(X, R) \subseteq \text{Čech}(X, R/\sqrt{2}).$$

In dimension 2, we get the $\sqrt{3}$-Lemma

$$\text{VR}(X, R) \subseteq \text{Čech}(X, R/\sqrt{3})$$

which relates the two forms of sensor network coverage discussed earlier.

Simplicial Homology

Goal

From a simplicial complex S, we wish to define a diagram of vector spaces and linear maps

$$0 \longleftarrow \mathrm{C}_0(S) \xleftarrow{\partial_0} \mathrm{C}_1(S) \xleftarrow{\partial_1} \mathrm{C}_2(S) \xleftarrow{\partial_2} \mathrm{C}_3(S) \xleftarrow{\partial_3} \cdots$$

satisfying $\partial^2 = 0$. This is the **simplicial chain complex** of S.

We also have the dual diagram

$$0 \longrightarrow \mathrm{C}^0(S) \xrightarrow{\delta^0} \mathrm{C}^1(S) \xrightarrow{\delta^1} \mathrm{C}^2(S) \xrightarrow{\delta^2} \mathrm{C}^3(S) \xrightarrow{\delta^3} \cdots$$

satisfying $\delta^2 = 0$. This is the **simplicial cochain complex** of S.

Homology

$\mathrm{Z}_k = \ker \partial_{k-1}, \quad \mathrm{B}_k = \mathrm{im}\, \partial_k, \quad \mathrm{H}_k = \mathrm{Z}_k / \mathrm{B}_k.$

Cohomology

$\mathrm{Z}^k = \ker \delta^k, \quad \mathrm{B}^k = \mathrm{im}\, \delta^{k-1}, \quad \mathrm{H}^k = \mathrm{Z}^k / \mathrm{B}^k.$

Simplicial Homology

Simplicial k-chains

We define $\mathrm{C}_k(S;\mathbb{F})$ in terms of generators and relations:

- a generator $[a_0, a_1, \ldots, a_k]$ whenever $\{a_0, a_1, \ldots, a_k\} \in S$,
- the relation $[a_0, a_1, \ldots, a_k] = 0$ if the a_i are not distinct;
- the relation

$$[a_0, a_1, \ldots, a_k] = (-1)^{\sigma}[a_{\sigma(0)}, a_{\sigma(1)}, \ldots, a_{\sigma(k)}]$$

  for any permutation σ.

Alphabetical convention

Given a total ordering of the vertices of S, the elements

$$[a_0, a_1, \ldots, a_k] \quad \text{such that} \quad a_0 < a_1 < \cdots < a_k \quad \text{and} \quad \{a_0, a_1, \ldots, a_k\} \in S$$

constiute a basis for C_k. Thus $\dim \mathrm{C}_k$ is equal to the number of k-simplices.

Simplicial Homology

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- a generator $[a_0, a_1, \dots, a_k]$ whenever $\{a_0, a_1, \dots, a_k\} \in S$,
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- the relation

$$[a_0, a_1, \dots, a_k] = (-1)^\sigma [a_{\sigma(0)}, a_{\sigma(1)}, \dots, a_{\sigma(k)}]$$

  for any permutation σ.

Boundary map

The boundary map $\partial_k : \mathrm{C}_{k+1} \to \mathrm{C}_k$ is defined on generators by

$$\partial_k[a_0, \dots, a_{k+1}] = \sum_{i=0}^{k+1} (-1)^i [\dots, \hat{a}_i, \dots]$$

where $[\dots, \hat{a}_i, \dots]$ denotes the result of deleting a_i from $[a_0, \dots, a_{k+1}]$.

The boundary maps are well-defined, and $\partial^2 = 0$.

Simplicial Cohomology

Simplicial k-cochains

We define $\mathrm{C}^k(S;\mathbb{F})$ to be the space of $\mathbb{F}$-valued functions

$$F(a_0, a_1, \dots, a_k)$$

defined for $(a_0, a_1, \dots, a_k)$ such that $\{a_0, a_1, \dots, a_k\} \in S$, and satisfying

- $F(a_0, a_1, \dots, a_k) = 0$ if the a_i are not distinct, and
- $F(a_0, a_1, \dots, a_k) = (-1)^\sigma F(a_{\sigma(0)}, a_{\sigma(1)}, \dots, a_{\sigma(k)})$ for any permutation σ.

Coboundary map

The coboundary map $\delta^k : \mathrm{C}^k \to \mathrm{C}^{k+1}$ is defined by the formula

$$\left[\delta^k F\right](a_0, \dots, a_{k+1}) = \sum_{i=0}^{k+1} (-1)^i F(\dots, \hat{a}_i, \dots)$$

where $(\dots, \hat{a}_i, \dots)$ denotes the result of deleting a_i from $(a_0, \dots, a_{k+1})$.

The coboundary maps are well-defined, and $\delta^2 = 0$.

Simplicial maps

Simplicial maps

Let S, T be simplicial complexes with respective vertex sets V, W.
A **simplicial map** $f : S \to T$ is a function

$$f : V \to W$$

such that $f\{a_0, a_1, \ldots, a_k\} \in T$ whenever $\{a_0, a_1, \ldots, a_k\} \in S$.

Write '$f\sigma \in T$ whenever $\sigma \in S$' for conciseness.

Induced chain map

A simplicial map $f : S \to T$ induces a family of maps

$$f_* : \mathrm{C}_k(S) \to \mathrm{C}_k(T)$$

by the formula $f_*[a_0, a_1, \ldots, a_k] = [f(a_0), f(a_1), \ldots, f(a_k)]$.

Induced homology map

Since $\partial f_* = f_* \partial$, this induces maps on homology

$$f_* : \mathrm{H}_k(S) \to \mathrm{H}_k(T).$$

Simplicial maps

Simplicial maps

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A **simplicial map** $f : S \to T$ is a function

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Write '$f\sigma \in T$ whenever $\sigma \in S$' for conciseness.

Induced cochain map

A simplicial map $f : S \to T$ induces a family of maps

$$f^* : \mathrm{C}^k(T) \to \mathrm{C}^k(S)$$

by the formula $\left[f_* F\right](a_0, a_1, \ldots, a_k) = F(f(a_0), f(a_1), \ldots, f(a_k))$.

Induced cohomology map

Since $f^*\delta = \delta f^*$, this induces maps on cohomology

$$f_* : \mathrm{H}^k(T) \to \mathrm{H}^k(S).$$

Simplicial maps

Contiguous maps

Two simplicial maps $f, g : S \to T$ are **contiguous** if

$$f\sigma \cup g\sigma \in T$$

whenever $\sigma \in S$.

Theorem

Contiguous simplicial maps $f, g : S \to T$ give rise to equal maps in homology

$$f_* = g_* : \mathrm{H}_k(S) \to \mathrm{H}_k(T)$$

and cohomology

$$f^* = g^* : \mathrm{H}^k(T) \to \mathrm{H}^k(S).$$

Persistent homology and cohomology

Philosophy of Topological Data Analysis

A finite data set X is given:

- metric space or similarity space; or
- subset of $\mathbb{R}^N$.

Construct a 1-parameter family of simplicial complexes $S(r)$ that 'approximate' X at different scales:

$$S(r_0) \xrightarrow{\subseteq} S(r_1) \xrightarrow{\subseteq} \cdots \xrightarrow{\subseteq} S(r_n)$$

Compute their homology

$$\mathrm{H}_k(S(r_0)) \longrightarrow \mathrm{H}_k(S(r_1)) \longrightarrow \cdots \longrightarrow \mathrm{H}_k(S(r_n))$$

or cohomology

$$\mathrm{H}^k(S(r_0)) \longleftarrow \mathrm{H}^k(S(r_1)) \longleftarrow \cdots \longleftarrow \mathrm{H}^k(S(r_n))$$

Invariants of the resulting diagrams are multiscale descriptors of the data.