2a. Cycles and Cocycles

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Geometry into Algebra

Today's Theme

Our goal today is to turn geometric objects into algebraic objects.

Also: geometric operations into algebraic operations.

Geometry into Algebra

Geometric simplices

Let $a, b, c \in \mathbb{R}^2$.

- $[a]$ is a 0-simplex, or point.
- $[a, b]$ is a 1-simplex, or directed edge.
- $[a, b, c]$ is a 2-simplex, or oriented triangle.

Supports

- $|[a]| = \{a\}$.
- $|[a, b]| = \{ta + ub \mid t, u \geq 0,\ t + u = 1\}$.
- $|[a, b, c]| = \{ta + ub + vc \mid t, u, v \geq 0,\ t + u + v = 1\}$.

Let $\mathcal{U} \subseteq \mathbb{R}^2$.

- $[a]$ is a 0-simplex of $\mathcal{U}$ if $|[a]| \subseteq \mathcal{U}$
- $[a, b]$ is a 1-simplex of $\mathcal{U}$ if $|[a, b]| \subseteq \mathcal{U}$
- $[a, b, c]$ is a 2-simplex of $\mathcal{U}$ if $|[a, b, c]| \subseteq \mathcal{U}$

Chain spaces

Let $\mathcal{U} \in \mathbb{R}^2$ and let $\mathbb{F}$ be a field. We define:

$\mathrm{C}_k(\mathcal{U}; \mathbb{F})$ = 'the vector space over $\mathbb{F}$ freely generated by the k-simplices of $\mathcal{U}$'

Elements of $\mathrm{C}_k = \mathrm{C}_k(\mathcal{U}) = \mathrm{C}_k(\mathcal{U}; \mathbb{F})$ are called k-chains.

Chains

Here is a typical 0-chain with its **geometric support**:

$$\alpha = \sum_{k=1}^{n} \lambda_k [a_k], \qquad |\alpha| = \{a_1, a_2, \ldots, a_k\}.$$

Here is a typical 1-chain with its geometric support:

$$\gamma = \sum_{k=1}^{n} \lambda_k [a_k, b_k], \qquad |\gamma| = \bigcup_{k=1}^{n} |[a_k, b_k]|.$$

Here is a typical 2-chain with its geometric support:

$$\sigma = \sum_{k=1}^{n} \lambda_k [a_k, b_k, c_k], \qquad |\sigma| = \bigcup_{k=1}^{n} |[a_k, b_k, c_k]|.$$

(In defining the geometric support, we assume that the k-simplices are distinct and the coefficients are non-zero.)

The boundary map ∂_0

We define a linear map

$$\mathrm{C}_0 \xleftarrow{\partial_0} \mathrm{C}_1$$

by setting

$$\partial_0[a, b] = [b] - [a]$$

on basis elements, and extending linearly to the whole of C_1.

Thus

$$\partial_0\big(\sum_k \lambda_k[a_k, b_k]\big) = \sum_k \lambda_k([b_k] - [a_k])$$

in general.

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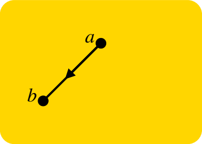

$$\partial([a, b]) = [b] - [a]$$

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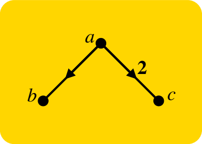

$$\partial([a,b]+2[a,c]) = -3[a]+[b]+2[c]$$

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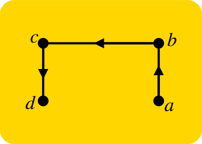

$$\partial([a, b] + [b, c] + [c, d]) = [d] - [a]$$

The boundary map ∂_0

We define a linear map

$$\mathrm{C}_0 \xleftarrow{\partial_0} \mathrm{C}_1$$

by setting

$$\partial_0[a, b] = [b] - [a]$$

on basis elements, and extending linearly to the whole of C_1.

Thus

$$\partial_0\left(\sum_k \lambda_k[a_k, b_k]\right) = \sum_k \lambda_k([b_k] - [a_k])$$

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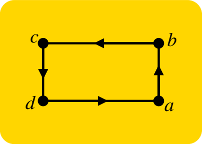

$$\partial([a, b] + [b, c] + [c, d] + [d, a]) = 0$$

Theorem

Let $\gamma \in \mathrm{C}_1(\mathcal{U};\mathbb{F})$. Then $\partial_0\gamma = 0$ if and only if γ is a 1-cycle over $\mathbb{F}$.

Proof

Taking the standard general form for γ, we have

$$\gamma = \textstyle\sum_k \lambda_k[a_k, b_k] \quad \Rightarrow \quad \partial_0\gamma = \textstyle\sum_k \lambda_k([b_k] - [a_k]).$$

Let $p \in \mathcal{U}$ be an arbitrary point. Then the coefficient of $[p]$ is

$$\textstyle\sum(\lambda_k \mid b_k = p) \;-\; \textstyle\sum(\lambda_k \mid a_k = p).$$

By definition, γ is a 1-cycle if and only if all of these terms are zero.

Cycle space

We write

$$\boxed{\mathrm{Z}_1(\mathcal{U};\mathbb{F}) = \ker(\partial_0 : \mathrm{C}_1(\mathcal{U};\mathbb{F}) \to \mathrm{C}_0(\mathcal{U};\mathbb{F}))}$$

We often abbreviate it $\mathrm{Z}_1 = \mathrm{Z}_1(\mathcal{U}) = \mathrm{Z}_1(\mathcal{U};\mathbb{F})$. It is a subspace of C_1.

The boundary map ∂_1

We define a linear map

$$\mathrm{C}_1 \xleftarrow{\partial_1} \mathrm{C}_2$$

by setting

$$\partial_1[a,b,c] = [b,c] - [a,c] + [a,b]$$

on basis elements, and extending linearly to the whole of C_2.

Fundamental fact: the boundary of a boundary is zero

In the sequence

$$\mathrm{C}_0 \xleftarrow{\partial_0} \mathrm{C}_1 \xleftarrow{\partial_1} \mathrm{C}_2$$

the composite linear map $\partial_0\partial_1$ is zero.

Proof

For any generator $[a,b,c]$ of C_2, we have

$$\partial_0\partial_1[a,b,c] = \partial_0([b,c] - [a,c] + [a,b]) = [c] - [b] - [c] + [a] + [b] - [a] = 0.$$

Since $\partial_0\partial_1$ is zero on the generators of C_2, it is zero on the whole space C_2.

Question

Which 1-chains γ lie in the image of ∂_1?

Boundary space

We write

$$\boxed{\mathrm{B}_1(\mathcal{U};\mathbb{F}) = \mathrm{im}(\partial_1 : \mathrm{C}_2(\mathcal{U};\mathbb{F}) \to \mathrm{C}_1(\mathcal{U};\mathbb{F}))}$$

We often abbreviate it $\mathrm{B}_1 = \mathrm{B}_1(\mathcal{U}) = \mathrm{B}_1(\mathcal{U};\mathbb{F})$. It is a subspace of B_1.

Necessary condition

γ must be a cycle.

Thus:

$$\mathrm{B}_1 \subseteq \mathrm{Z}_1 \subseteq \mathrm{C}_1$$

Proof

If $\gamma = \partial_1\sigma$ then $\partial_0\gamma = \partial_0\partial_1\sigma = 0$ so γ is a cycle.

Geometry into Algebra

Question

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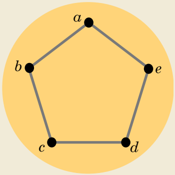
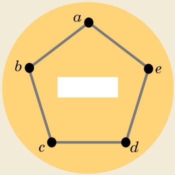
Is the necessary condition sufficient?

Let $\gamma = [a, b] + [b, c] + [c, d] + [d, e] - [a, e]$ in these two domains:

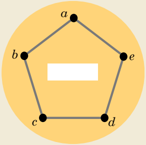




Is γ a boundary?

Theorem

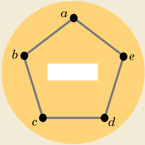
The 1-cycle $\gamma = [a, b] + [b, c] + [c, d] + [d, e] - [a, e]$ in the following domain



is not a 1-boundary.

Theorem

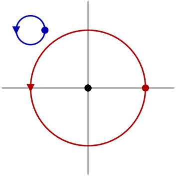
The 1-cycle $\gamma = [a, b] + [b, c] + [c, d] + [d, e] - [a, e]$ in the following domain



is not a 1-boundary.

Hint

How did we show that these two loops were not homotopic?



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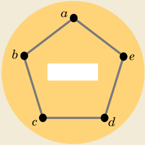

is not a 1-boundary.

Proof

Define $T : \mathrm{C}_1(\mathcal{U}; \mathbb{F}) \to \mathbb{F}$ as follows. Let p be a point in the rectangle. Define

$$T(\zeta) = \mathrm{w}_{\mathbb{F}}(\zeta, p).$$

Then

- $T\partial_1[a, b, c] = 0$ for any $[a, b, c]$ with $p \notin |[a, b, c]|$.
- $T\partial_1$ is the zero map on $\mathrm{C}_2(\mathcal{U}; \mathbb{F})$.
- $T(\gamma) = 1$.

Thus we cannot solve $\gamma = \partial_1\sigma$.

Coverage Theorem

Suppose $\gamma = \partial_1 \sigma$, where $\gamma \in \mathrm{Z}_1(\mathbb{R}^2)$ and $\sigma \in \mathrm{C}_2(\mathbb{R}^2)$.
Then $|\sigma|$ contains all points $p \in \mathbb{R}^2 - |\gamma|$ for which $\mathrm{w}(\gamma, p) \neq 0$.

Proof

Suppose we have a solution

$$\sigma = \sum \lambda_k [a_k, b_k, c_k]$$

where p does not belong to any $|[a_k, b_k, c_k]|$. Then $\mathrm{w}(\gamma, p) = \mathrm{w}(\partial_1 \sigma, p) = 0$.

Another point of view

An **obstruction** to the truth of the equation

$$\mathrm{Z}_1(\mathcal{U}) \stackrel{?}{=} \mathrm{B}_1(\mathcal{U})$$

is the existence of a cycle $\gamma \in \mathrm{Z}_1(\mathcal{U})$ and a point $p \in \mathbb{R}^2 - \mathcal{U}$ with $\mathrm{w}(\gamma, p) \neq 0$.

The quotient vector space $\mathrm{H}_1(\mathcal{U}) = \mathrm{Z}_1(\mathcal{U})/\mathrm{B}_1(\mathcal{U})$ tells us something about 'enclosable holes' in $\mathcal{U}$.