1. Winding Numbers

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Goal

Continuous winding number

We seek a function

\[ w : \text{Loops}(\mathbb{R}^2 - \{0\}) \rightarrow \mathbb{Z} \]

that counts the number of times the loop winds around 0.

[Examples on board]

- Loops are oriented.
- Counterclockwise = positive.

Approaches

- Covering spaces, fundamental group, homotopy theory
- Contour integration
- Discrete approximations, homology theory
Argument function

First attempt

\[ \text{arg} : \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R} \]

This measures the angle of a point \( a \neq 0 \) from the positive \( x \)-axis.

[Example on board]

Corrected version

\[ \text{arg} : \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R}/2\pi\mathbb{Z} \]

Write \([\theta]\) for the equivalence class of \( \theta \) modulo \( 2\pi \).

Explicit formulas

\[
\text{arg}((x, y)) = \begin{cases} 
[\arctan(y/x)] & \text{if } x > 0 \\
[\arctan(y/x) + \pi] & \text{if } x < 0 \\
[-\arctan(x/y) + \frac{\pi}{2}] & \text{if } y > 0 \\
[-\arctan(x/y) - \frac{\pi}{2}] & \text{if } y < 0 
\end{cases}
\]

These are continuous and agree where their domains overlap.
Lifts of the argument function

It is useful to consider functions

\[ \overline{\text{arg}} : \mathcal{U} \rightarrow \mathbb{R} \]

that satisfy \( \overline{\text{arg}}(a) = \text{arg}(a) \) for all \( a \in \mathcal{U} \).

At most one of the following can be satisfied:
- \( \mathcal{U} = \mathbb{R}^2 - \{0\} \)
- \( \overline{\text{arg}} \) is continuous

The lift from a ray

Let

\[ \overline{\text{arg}}_\phi : \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R} \]

be the unique lift taking values in \([\phi, \phi + 2\pi)\).

The lift \( \overline{\text{arg}}_\phi \) is continuous except on the ray

\[ R_\phi = \text{arg}^{-1}(\lceil \phi \rceil). \]
### Angle cocycle

#### Line segments

Let \( a, b \in \mathbb{R}^2 \).

\[
[a, b] = \text{‘directed line segment from } a \text{ to } b \text{’}
\]

\[
|[a, b]| = \{ta + ub | t, u \geq 0, t + u = 1\}
\]

= set of points which lie on \([a, b]\)

#### Angle cocycle

Let \( a, b \in \mathbb{R}^2 \) such that \( 0 \notin |[a, b]| \).

\[
\Lambda([a, b], 0) = \text{‘angle subtended at } 0 \text{ by } [a, b] \text{’}
\]

[Example on board]

#### Formally

Let \( \Lambda([a, b], 0) \) be the unique real number \( \theta \) such that:

- \(-\pi < \theta < \pi\)
- \([\theta] = \text{arg}(b) - \text{arg}(a)\)
### Angle cocycle

#### Line segments

Let \( a, b \in \mathbb{R}^2 \).

\[
[a, b] = \{ta + ub \mid t, u \geq 0, t + u = 1\}
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#### Angle cocycle

Let \( a, b, p \in \mathbb{R}^2 \) such that \( p \notin [a, b] \).

\[
\Lambda([a, b], p) = \text{‘angle subtended at } p \text{ by } [a, b]' \]

[Example on board]

#### Formally

Let \( \Lambda([a, b], p) \) be the unique real number \( \theta \) such that:

- \(-\pi < \theta < \pi\)
- \([\theta] = \arg(b - p) - \arg(a - p)\)
Theorem

\( \Lambda(a, b, p) \) is continuous as a function of \( a, b, p \) (on its domain of definition).

Proof

Regarding \( a, b, p \) as complex numbers, one can show that

\[
\Lambda([a, b], p) = \text{arg}_{-\pi} \left( \frac{b - p}{a - p} \right).
\]

This is a composite of continuous functions, since

\[
\text{arg}_{-\pi} (x + iy) = 2 \arctan \left( \frac{y}{x + \sqrt{x^2 + y^2}} \right).
\]
Closed polygons

Let \( \gamma = P(a_0, a_1, \ldots, a_n) \) denote the ‘directed polygon with vertices \( a_0, a_1, \ldots, a_n \) and edges \([a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n]\)’. 

It is **closed** if \( a_0 = a_n \). Its **support** is 

\[
|\gamma| = \bigcup_{j=1}^{n} |[a_{j-1}, a_j]|
\]

We often write 

\[
\gamma = [a_0, a_1] + [a_1, a_2] + \cdots + [a_{n-1}, a_n] = \sum_{j=1}^{n} [a_{j-1}, a_j]
\]
The winding number of a closed polygon

### Winding number

Let $\gamma = P(a_0, a_1, \ldots, a_n)$ be closed, and let $p \notin |\gamma|$. We define:

$$w(\gamma, p) = \frac{1}{2\pi} \sum_{j=1}^{n} \Lambda([a_{j-1}, a_j], p)$$

[Examples on board]

### Properties of the winding number

- $w(\gamma, p)$ is an integer.
- $w(\gamma, p)$ is constant on each component of $\mathbb{R}^2 - |\gamma|$.
- $w(\gamma, p)$ is equal to the number of times $\gamma$ crosses a generic ray from $p$.
- $w(\gamma, p) = 0$ if $\min_j |a_j - p| > \max_{j,k} |a_j - a_k|$.
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Proof

We may assume $p = 0$. Let $\overline{\arg}$ be any lift of $\arg$. Then:

$$\Lambda([a_{j-1}, a_j], 0) = [\overline{\arg}(a_j) - \overline{\arg}(a_{j-1})]$$

in $\mathbb{R}/2\pi \mathbb{Z}$, so

$$\Lambda([a_{j-1}, a_j], 0) = \overline{\arg}(a_j) - \overline{\arg}(a_{j-1}) + 2\pi m_j$$

for some integers $m_j$. In the sum, the $\overline{\arg}$ terms cancel and we get

$$w(\gamma, 0) = \frac{1}{2\pi} \sum_{j=1}^{n} \Lambda([a_{j-1}, a_j], 0) = m_1 + m_2 + \cdots + m_n.$$
Properties of the winding number

- $w(\gamma, p)$ is an integer.
- $w(\gamma, p)$ is constant on each component of $\mathbb{R}^2 - |\gamma|$.
- $w(\gamma, p)$ is equal to the number of times $\gamma$ crosses a generic ray from $p$.
- $w(\gamma, p) = 0$ if $\min_j |a_j - p| > \max_{j,k} |a_j - a_k|$.

Proof

Each term $\Lambda([a_{j-1}, a_j], p)$ is continuous on $\mathbb{R}^2 - |[a_{j-1}, a_j]|$ and hence on $\mathbb{R}^2 - |\gamma|$. Thus $\gamma$ is continuous. Since it is integer-valued, it must be constant on each component.
Properties of the winding number

- \( w(\gamma, p) \) is an integer.
- \( w(\gamma, p) \) is constant on each component of \( \mathbb{R}^2 - |\gamma| \).
- \( w(\gamma, p) \) is equal to the number of times \( \gamma \) crosses a generic ray from \( p \).
- \( w(\gamma, p) = 0 \) if \( \min_j |a_j - p| > \max_{j,k} |a_j - a_k| \).

[Example on board]

It is easiest to assume that the ray avoids the vertices of \( \gamma \).

Proof

We may assume \( p = 0 \). Let \( \overline{\arg}_\phi \) be the lift from the ray \( R_\phi \). Then

\[
\Lambda([a_{j-1}, a_j], 0) = \overline{\arg}_\phi(a_j) - \overline{\arg}_\phi(a_{j-1}) + 2\pi \xi_j
\]

where \( \xi_j \in \{0, \pm 1\} \) is the crossing number of the edge across the ray. Then

\[
w(\gamma, p) = \frac{1}{2\pi} \sum_{j=1}^{n} \Lambda([a_{j-1}, a_j], 0) = \xi_1 + \xi_2 + \cdots + \xi_n.
\]
### Properties of the winding number

- \( w(\gamma, p) \) is an integer.
- \( w(\gamma, p) \) is constant on each component of \( \mathbb{R}^2 - |\gamma| \).
- \( w(\gamma, p) \) is equal to the number of times \( \gamma \) crosses a generic ray from \( p \).
- \( w(\gamma, p) = 0 \) if \( \min_j |a_j - p| > \max_{j,k} |a_j - a_k| \).

### Proof

The inequalities confine \( \gamma \) to a disk centered at \( a_0 \) that avoids \( p \). Pick a ray that avoids this disk.

Let’s call this the **Distant Cycle Lemma**.
Generalizations

Integer 1-cycles

An integer 1-cycle is a formal sum of directed edges

\[ \gamma = \sum_{k=1}^{n} [a_k, b_k] \]

such that each \( p \in \mathbb{R}^2 \) occurs equally often in the lists \((a_k)\) and \((b_k)\).

[Example on board]

\[ w(\gamma, p) = \frac{1}{2\pi} \sum_{j=1}^{n} \Lambda([a_{j-1}, a_j], p) \]

Properties of the winding number

- \( w(\gamma, p) \) is additive in \( \gamma \).
- \( w(\gamma, p) \) is an integer.
- \( w(\gamma, p) \) is constant on each component of \( \mathbb{R}^2 - |\gamma| \).
- \( w(\gamma, p) \) is equal to the number of times \( \gamma \) crosses a generic ray from \( p \).
- \( w(\gamma, p) = 0 \) if \( \min_j |a_j - p| > \max_{j,k} |a_j - a_k| \).
1-cycles in a ring $\mathbb{A}$

A 1-cycle in $\mathbb{A}$ is a formal linear combination of directed edges

$$\gamma = \sum_{k=1}^{n} \lambda_k [a_k, b_k]$$

such that $\sum (\lambda_k \mid a_k = p) = \sum (\lambda_k \mid b_k = p)$ for each $p \in \mathbb{R}^2$.

[Example on board]

$$w(\gamma, p) = \sum_{j=1}^{n} \lambda_k \xi([a_{j-1}, a_j], R_{\phi, p})$$

Properties of the winding number

- $w(\gamma, p)$ is linear in $\gamma$.
- $w(\gamma, p)$ is an element of $\mathbb{A}$.
- $w(\gamma, p)$ is constant on each component of $\mathbb{R}^2 - |\gamma|$.
- $w(\gamma, p)$ is independent of the choice of generic ray from $p$.
- $w(\gamma, p) = 0$ if $\min_j |a_j - p| > \max_{j,k} |a_j - a_k|$.
1. Winding Numbers
The space of loops

Let $U \subseteq \mathbb{R}^2$. Then $\text{Loops}(U)$ is the set of continuous maps

$$f : [0, 1] \longrightarrow U$$

such that $f(0) = f(1)$. 
Continuous loops

**Homotopy of loops**

Two loops \( f, g \in \text{Loops}(U) \) are **homotopic** if there exists a continuous map

\[
H : [0, 1] \times [0, 1] \to U
\]
on the unit square, such that

- \( H(0, t) = f(t) \) for all \( t \in [0, 1] \),
- \( H(1, t) = g(t) \) for all \( t \in [0, 1] \),
- \( H(s, 0) = H(s, 1) \) for all \( s \in [0, 1] \).

\( H \) is called a **homotopy** between \( f \) and \( g \). We write \( f \simeq g \) or \( f \simeq_H g \).

Homotopy ‘\( \simeq \)’ is an equivalence relation.
Theorem

Let $\mathcal{U} \subseteq \mathbb{R}^2$ be convex. Then any two loops $f, g \in \text{Loops}(\mathcal{U})$ are homotopic.

Proof

We have $f \simeq_H g$ via the **straight-line homotopy**

$$H(s, t) = (1 - s)f(t) + sg(t).$$

Convexity implies that this is entirely contained in $\mathcal{U}$.

[Example on board]
Continuous loops

Theorem

There are loops in $\mathbb{R}^2 - \{0\}$ which are not homotopic to each other.

How do we prove that they are not homotopic in $\mathbb{R}^2 - \{0\}$?
The continuous winding number

Theorem

There exists a function \( \text{Loops}(\mathbb{R}^2 - \{0\}) \to \mathbb{Z} \), expressed by the notation

\[
f \mapsto w(f, 0)
\]

and called ‘the winding number of \( f \) around \( 0 \)’, such that

- if \( f \sim g \) in \( \text{Loops}(\mathbb{R}^2 - \{0\}) \) then \( w(f, 0) = w(g, 0) \); and
- if \( f = e^{2\pi i t} \) then \( w(f, 0) = d \).

Define \( w(f, p) \) by translation, for any \( p \in \mathbb{R}^2 \).
The continuous winding number

**Theorem**

There are loops in $\mathbb{R}^2 - \{0\}$ which are not homotopic to each other.

How do we prove that they are not homotopic in $\mathbb{R}^2 - \{0\}$?

- The blue loop is homotopic to $e^{2\pi i 0 t}$ and therefore has winding number 0.
- The red loop is equal to $e^{2\pi i 1 t}$ and therefore has winding number 1.
Subdivide the interval by \( T = (t_0, t_1, \ldots, t_n) \), where
\[
0 = t_0 < t_1 < \cdots < t_n = 1.
\]
Consider the polygonal approximation
\[
f_T = \sum_{k=1}^{n} [f(t_{k-1}), f(t_k)]
\]
Define
\[
w(f, 0) = w(f_T, 0).
\]
The continuous winding number: well-defined

Is it well-defined?

Let $M = \min_t |f(t)|$. Let $\delta > 0$ such that $|t - t'| < \delta$ implies $|f(t) - f(t')| < M$. Use subdivisions $T$ such that $\max_k |t_k - t_{k-1}| < \delta$.

- It follows that the edges of $T$ avoid $0$.
- Adding further points does not change the winding number.

We used:

- Additivity of the winding number for 1-cycles.
- Distant Cycle Lemma.
The continuous winding number: homotopy invariance

What happens when \( f \simeq_H g \)?

Let \( M = \min_{s,t} |H(s, t)| \).

Let \( \delta > 0 \) such that \( |s - s'|, |t - t'| < \delta \) implies \( |H(s, t) - H(s', t')| < M \).

Divide the square into small squares of side less than \( \delta \).

\( H \) maps each small square to a quadrilateral with winding number 0.

It follows that \( 0 = w(g_T, 0) - w(f_T, 0) = w(g, 0) - w(f, 0) \).

We used:

- Additivity of the winding number for 1-cycles.
- Distant Cycle Lemma.

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1. Winding Numbers
What happens when $f = e^{2\pi i dt}$?

Let

$$T = (0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1)$$

where $N > 2|d|$. Then

$$w(e^{2\pi i dt}, 0) = w([e^{2\pi i dt}]_T, 0) = \frac{1}{2\pi} \sum_{k=1}^{N} \frac{2\pi d}{N} = d$$

as required.
Application: Brouwer’s Fixed Point Theorem

Theorem
Every continuous map \( p : D^2 \to D^2 \) has a fixed point.
(Here \( D^2 \) is the closed unit disk in the plane.)

More generally
We may replace \( D^2 \) by any space that is topologically equivalent.
Application: Brouwer’s Fixed Point Theorem

**Theorem**

Every continuous map \( p : D^2 \rightarrow D^2 \) has a fixed point. (Here \( D^2 \) is the closed unit disk in the plane.)

**More generally**

The theorem fails for the open disk.
Application: Brouwer’s Fixed Point Theorem

**Theorem**

Every continuous map $p : D^2 \to D^2$ has a fixed point. (Here $D^2$ is the closed unit disk in the plane.)

**More generally**

The theorem fails for the annulus.
Theorem

Every continuous map \( p : D^2 \rightarrow D^2 \) has a fixed point.
(Here \( D^2 \) is the closed unit disk in the plane.)

Proof

Suppose \( p \) has no fixed points. Then \( q(x) = x - p(x) \) is never zero.

- Let \( f(t) = e^{2\pi it} \).
  \( w(f, 0) = 1 \)
- Let \( J(s, t) = e^{2\pi it} - sp(e^{2\pi it}) \).
  \( w(f, 0) = w(g, 0) \)
- Let \( g(t) = q(e^{2\pi it}) \).
  \( w(g, 0) =? \)
- Let \( K(s, t) = q(se^{2\pi it}) \).
  \( w(g, 0) = w(h, 0) \)
- Let \( h(t) = q(0) \).
  \( w(h, 0) = 0 \)
1. Winding Numbers
Application: Fundamental Theorem of Algebra

**Theorem**

Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has $d$ roots.)

**Proof**

Suppose $p$ has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. Then $w(f, 0) = d$.
- Let $H(s, t) = p(sRe^{2\pi it})$. Then $w(f, 0) = w(g, 0)$.
- Let $g(t) = p(0)$. Then $w(g, 0) = 0$. 

Vin de Silva Pomona College 1. Winding Numbers
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Proof

Suppose \( p \) has no roots. Let \( R > 0 \) be very large.

- Let \( f(t) = p(Re^{2\pi it}) \). \hspace{1cm} w(f, 0) = d
- Let \( H(s, t) = p(sRe^{2\pi it}) \). \hspace{1cm} w(f, 0) = w(g, 0)
- Let \( g(t) = p(0) \). \hspace{1cm} w(g, 0) = 0

\[ z^3 + (2-i)z^2 + iz + 2 \]
Theorem

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1. Winding Numbers
Application: Fundamental Theorem of Algebra

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Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

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Proof

Suppose $p$ has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $w(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $w(f, 0) = w(g, 0)$
- Let $g(t) = p(0)$. $w(g, 0) = 0$
Application: Fundamental Theorem of Algebra

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**Proof**

Suppose $p$ has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$.  \hspace{2cm}  w(f, 0) = d
- Let $H(s, t) = p(sRe^{2\pi it})$.  \hspace{2cm}  w(f, 0) = w(g, 0)
- Let $g(t) = p(0)$.  \hspace{2cm}  w(g, 0) = 0

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1. Winding Numbers
Theorem
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.
(By repeated factorization, it therefore has $d$ roots.)

Proof
Suppose $p$ has no roots. Let $R > 0$ be very large.
- Let $f(t) = p(Re^{2\pi it})$. $w(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $w(f, 0) = w(g, 0)$
- Let $g(t) = p(0)$. $w(g, 0) = 0$