1. Winding Numbers

Vin de Silva
Pomona College

CBMS Lecture Series, Macalester College, June 2017

Goal

Continuous winding number

We seek a function

$$w : \mathrm{Loops}(\mathbb{R}^2 - \{0\}) \longrightarrow \mathbb{Z}$$

that counts the number of times the loop winds around 0.

[Examples on board]

- Loops are oriented.
- Counterclockwise = positive.

Approaches

- Covering spaces, fundamental group, homotopy theory
- Contour integration
- Discrete approximations, homology theory

Argument function

First attempt

$$\arg : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}$$

This measures the angle of a point $a \neq 0$ from the positive x-axis.

[Example on board]

Corrected version

$$\arg : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$$

Write $[\theta]$ for the equivalence class of θ modulo 2π.

Explicit formulas

$$\arg((x,y)) = \begin{cases} [\arctan(y/x)] & \text{if } x > 0 \\ [\arctan(y/x) + \pi] & \text{if } x < 0 \\ [-\arctan(x/y) + \frac{\pi}{2}] & \text{if } y > 0 \\ [-\arctan(x/y) - \frac{\pi}{2}] & \text{if } y < 0 \end{cases}$$

These are continuous and agree where their domains overlap.

Lifts of the argument function

Lifts

It is useful to consider functions

$$\overline{\arg} : \mathcal{U} \longrightarrow \mathbb{R}$$

that satisfy $\left[\overline{\arg}(a)\right] = \arg(a)$ for all $a \in \mathcal{U}$.

At most one of the following can be satisfied:

- $\mathcal{U} = \mathbb{R}^2 - \{0\}$
- $\overline{\arg}$ is continuous

The lift from a ray

Let

$$\overline{\arg}_\phi : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}$$

be the unique lift taking values in $[\phi, \phi + 2\pi)$.

The lift $\overline{\arg}_\phi$ is continuous except on the ray

$$R_\phi = \arg^{-1}([\phi]).$$

Line segments

Let $a, b \in \mathbb{R}^2$.

$$\begin{aligned}[a, b] &= \text{‘directed line segment from } a \text{ to } b\text{’} \\ |[a, b]| &= \{ta + ub \mid t, u \geq 0,\ t + u = 1\} \\ &= \text{set of points which lie on } [a, b]\end{aligned}$$

Angle cocycle

Let $a, b \in \mathbb{R}^2$ such that $0 \notin |[a, b]|$.

$$\Lambda([a, b], 0) = \text{‘angle subtended at 0 by } [a, b]\text{’}$$

[Example on board]

Formally

Let $\Lambda([a, b], 0)$ be the unique real number θ such that:

- $-\pi < \theta < \pi$
- $[\theta] = \arg(b) - \arg(a)$

Angle cocycle

Line segments

Let $a, b \in \mathbb{R}^2$.

$$\begin{aligned} [a, b] &= \text{`directed line segment from } a \text{ to } b\text{'} \\ |[a, b]| &= \{ta + ub \mid t, u \geq 0,\ t + u = 1\} \\ &= \text{set of points which lie on } [a, b] \end{aligned}$$

Angle cocycle

Let $a, b, p \in \mathbb{R}^2$ such that $p \notin |[a, b]|$.

$$\Lambda([a, b], p) = \text{`angle subtended at } p \text{ by } [a, b]\text{'}$$

[Example on board]

Formally

Let $\Lambda([a, b], p)$ be the unique real number θ such that:

- $-\pi < \theta < \pi$
- $[\theta] = \arg(b - p) - \arg(a - p)$

Continuity

Theorem

$\Lambda(a, b, p)$ is continuous as a function of a, b, p (on its domain of definition).

Proof

Regarding a, b, p as complex numbers, one can show that

$$\Lambda([a, b], p) = \overline{\arg}_{-\pi}\left(\frac{b-p}{a-p}\right).$$

This is a composite of continuous functions, since

$$\overline{\arg}_{-\pi}(x + iy) = 2\arctan\left(\frac{y}{x + \sqrt{x^2 + y^2}}\right).$$

Closed polygons

Closed polygons

Let $\gamma = P(a_0, a_1, \ldots, a_n)$ denote the 'directed polygon with vertices $a_0, a_1, \ldots, a_n$ and edges $[a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n]$'.

It is **closed** if $a_0 = a_n$. Its **support** is

$$|\gamma| = \bigcup_{j=1}^{n} |[a_{j-1}, a_j]|$$

We often write

$$\gamma = [a_0, a_1] + [a_1, a_2] + \cdots + [a_{n-1}, a_n] = \sum_{j=1}^{n} [a_{j-1}, a_j]$$

The winding number of a closed polygon

Winding number

Let $\gamma = P(a_0, a_1, \ldots, a_n)$ be closed, and let $p \notin |\gamma|$. We define:

$$\mathrm{w}(\gamma, p) = \frac{1}{2\pi} \sum_{j=1}^{n} \Lambda([a_{j-1}, a_j], p)$$

[Examples on board]

Properties of the winding number

- $\mathrm{w}(\gamma, p)$ is an integer.
- $\mathrm{w}(\gamma, p)$ is constant on each component of $\mathbb{R}^2 - |\gamma|$.
- $\mathrm{w}(\gamma, p)$ is equal to the number of times γ crosses a generic ray from p.
- $\mathrm{w}(\gamma, p) = 0$ if $\min_j |a_j - p| > \max_{j,k} |a_j - a_k|$.

Properties of the winding number

- $\mathrm{w}(\gamma, p)$ is an integer.
- $\mathrm{w}(\gamma, p)$ is constant on each component of $\mathbb{R}^2 - |\gamma|$.
- $\mathrm{w}(\gamma, p)$ is equal to the number of times γ crosses a generic ray from p.
- $\mathrm{w}(\gamma, p) = 0$ if $\min_j |a_j - p| > \max_{j,k} |a_j - a_k|$.

Proof

We may assume $p = 0$. Let $\overline{\arg}$ be any lift of arg. Then:

$$\left[\Lambda([a_{j-1}, a_j], 0)\right] = \left[\overline{\arg}(a_j) - \overline{\arg}(a_{j-1})\right]$$

in $\mathbb{R}/2\pi\mathbb{Z}$, so

$$\Lambda([a_{j-1}, a_j], 0) = \overline{\arg}(a_j) - \overline{\arg}(a_{j-1}) + 2\pi m_j$$

for some integers m_j. In the sum, the $\overline{\arg}$ terms cancel and we get

$$\mathrm{w}(\gamma, 0) = \frac{1}{2\pi} \sum_{j=1}^{n} \Lambda([a_{j-1}, a_j], 0) = m_1 + m_2 + \cdots + m_n.$$

Properties of the winding number

- $\mathrm{w}(\gamma, p)$ is an integer.
- $\mathrm{w}(\gamma, p)$ is constant on each component of $\mathbb{R}^2 - |\gamma|$.
- $\mathrm{w}(\gamma, p)$ is equal to the number of times γ crosses a generic ray from p.
- $\mathrm{w}(\gamma, p) = 0$ if $\min_j |a_j - p| > \max_{j,k} |a_j - a_k|$.

Proof

Each term $\Lambda([a_{j-1}, a_j], p)$ is continuous on $\mathbb{R}^2 - |[a_{j-1}, a_j]|$ and hence on $\mathbb{R}^2 - |\gamma|$. Thus γ is continuous. Since it is integer-valued, it must be constant on each component.

Properties of the winding number

- $\mathrm{w}(\gamma, p)$ is an integer.
- $\mathrm{w}(\gamma, p)$ is constant on each component of $\mathbb{R}^2 - |\gamma|$.
- $\mathrm{w}(\gamma, p)$ is equal to the number of times γ crosses a generic ray from p.
- $\mathrm{w}(\gamma, p) = 0$ if $\min_j |a_j - p| > \max_{j,k} |a_j - a_k|$.

[Example on board]

It is easiest to assume that the ray avoids the vertices of γ.

Proof

We may assume $p = 0$. Let $\overline{\arg}_\phi$ be the lift from the ray R_ϕ. Then

$$\Lambda([a_{j-1}, a_j], 0) = \overline{\arg}_\phi(a_j) - \overline{\arg}_\phi(a_{j-1}) + 2\pi\xi_j$$

where $\xi_j \in \{0, \pm 1\}$ is the **crossing number** of the edge across the ray. Then

$$\mathrm{w}(\gamma, p) = \frac{1}{2\pi} \sum_{j=1}^{n} \Lambda([a_{j-1}, a_j], 0) = \xi_1 + \xi_2 + \cdots + \xi_n.$$

Properties of the winding number

- $\mathrm{w}(\gamma, p)$ is an integer.
- $\mathrm{w}(\gamma, p)$ is constant on each component of $\mathbb{R}^2 - |\gamma|$.
- $\mathrm{w}(\gamma, p)$ is equal to the number of times γ crosses a generic ray from p.
- $\mathrm{w}(\gamma, p) = 0$ if $\min_j |a_j - p| > \max_{j,k} |a_j - a_k|$.

[Example on board]

Proof

The inequalities confine γ to a disk centered at a_0 that avoids p. Pick a ray that avoids this disk.

Let's call this the **Distant Cycle Lemma**.

Generalizations

Integer 1-cycles

An integer 1-cycle is a formal sum of directed edges

$$\gamma = \sum_{k=1}^{n} [a_k, b_k]$$

such that each $p \in \mathbb{R}^2$ occurs equally often in the lists (a_k) and (b_k).

[Example on board]

$$\mathrm{w}(\gamma, p) = \frac{1}{2\pi} \sum_{j=1}^{n} \Lambda([a_{j-1}, a_j], p)$$

Properties of the winding number

- $\mathrm{w}(\gamma, p)$ is additive in γ.
- $\mathrm{w}(\gamma, p)$ is an integer.
- $\mathrm{w}(\gamma, p)$ is constant on each component of $\mathbb{R}^2 - |\gamma|$.
- $\mathrm{w}(\gamma, p)$ is equal to the number of times γ crosses a generic ray from p.
- $\mathrm{w}(\gamma, p) = 0$ if $\min_j |a_j - p| > \max_{j,k} |a_j - a_k|$.

Generalizations

1-cycles in a ring $\mathbb{A}$

A 1-cycle in $\mathbb{A}$ is a formal linear combination of directed edges

$$\gamma = \sum_{k=1}^{n} \lambda_k [a_k, b_k]$$

such that $\sum (\lambda_k \mid a_k = p) = \sum (\lambda_k \mid b_k = p)$ for each $p \in \mathbb{R}^2$.

[Example on board]

$$\mathrm{w}(\gamma, p) = \textstyle\sum_{j=1}^{n} \lambda_k \, \xi([a_{j-1}, a_j], R_{\phi,p})$$

Properties of the winding number

- $\mathrm{w}(\gamma, p)$ is linear in γ.
- $\mathrm{w}(\gamma, p)$ is an element of $\mathbb{A}$.
- $\mathrm{w}(\gamma, p)$ is constant on each component of $\mathbb{R}^2 - |\gamma|$.
- $\mathrm{w}(\gamma, p)$ is independent of the choice of generic ray from p.
- $\mathrm{w}(\gamma, p) = 0$ if $\min_j |a_j - p| > \max_{j,k} |a_j - a_k|$.

Continuous loops

The space of loops

Let $\mathcal{U} \subseteq \mathbb{R}^2$. Then Loops($\mathcal{U}$) is the set of continuous maps

$$f : [0,1] \longrightarrow \mathcal{U}$$

such that $f(0) = f(1)$.

Homotopy of loops

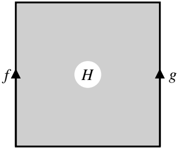
Two loops $f, g \in \mathrm{Loops}(\mathcal{U})$ are **homotopic** if there exists a continuous map

$$H : [0,1] \times [0,1] \longrightarrow \mathcal{U}$$

on the unit square, such that

- $H(0,t) = f(t)$ for all $t \in [0,1]$,
- $H(1,t) = g(t)$ for all $t \in [0,1]$,
- $H(s,0) = H(s,1)$ for all $s \in [0,1]$.

H is called a **homotopy** between f and g. We write $f \simeq g$ or $f \simeq_H g$.



Homotopy '$\simeq$' is an equivalence relation.

Continuous loops

Theorem

Let $\mathcal{U} \subseteq \mathbb{R}^2$ be convex. Then any two loops $f, g \in \text{Loops}(\mathcal{U})$ are homotopic.

Proof

We have $f \simeq_H g$ via the **straight-line homotopy**

$$H(s,t) = (1-s)f(t) + sg(t).$$

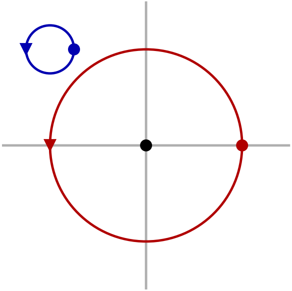
Convexity implies that this is entirely contained in $\mathcal{U}$.

[Example on board]

Continuous loops

Theorem

There are loops in $\mathbb{R}^2 - \{0\}$ which are not homotopic to each other.



How do we prove that they are not homotopic in $\mathbb{R}^2 - \{0\}$?

The continuous winding number

Theorem

There exists a function $\text{Loops}(\mathbb{R}^2 - \{0\}) \to \mathbb{Z}$, expressed by the notation

$$f \mapsto \text{w}(f, 0)$$

and called 'the winding number of f around 0', such that

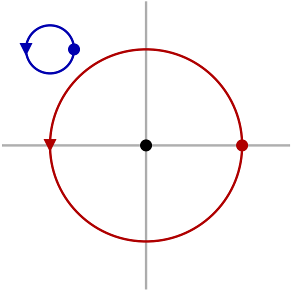
- if $f \simeq g$ in $\text{Loops}(\mathbb{R}^2 - \{0\})$ then $\text{w}(f, 0) = \text{w}(g, 0)$; and
- if $f = e^{2\pi i d t}$ then $\text{w}(f, 0) = d$.

Define $\text{w}(f, p)$ by translation, for any $p \in \mathbb{R}^2$.

The continuous winding number

Theorem

There are loops in $\mathbb{R}^2 - \{0\}$ which are not homotopic to each other.



How do we prove that they are not homotopic in $\mathbb{R}^2 - \{0\}$?

- The blue loop is homotopic to $e^{2\pi i 0 t}$ and therefore has winding number 0.
- The red loop is equal to $e^{2\pi i 1 t}$ and therefore has winding number 1.

The continuous winding number: construction

Construction

Subdivide the interval by $T = (t_0, t_1, \ldots, t_n)$, where

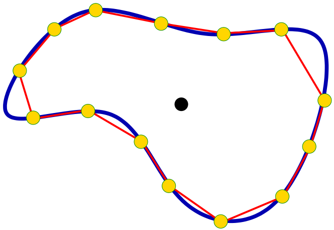
$$0 = t_0 < t_1 < \cdots < t_n = 1.$$

Consider the polygonal approximation

$$f_T = \sum_{k=1}^{n} \left[f(t_{k-1}), f(t_k) \right]$$

Define

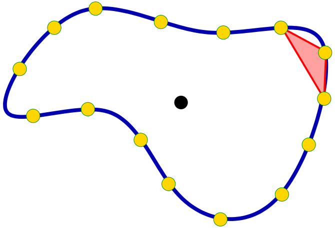
$$\mathrm{w}(f, 0) = \mathrm{w}(f_T, 0).$$



Is it well-defined?

Let $M = \min_t |f(t)|$. Let $\delta > 0$ such that $|t - t'| < \delta$ implies $|f(t) - f(t')| < M$. Use subdivisions T such that $\max_k |t_k - t_{k-1}| < \delta$.

- It follows that the edges of T avoid 0.
- Adding further points does not change the winding number.

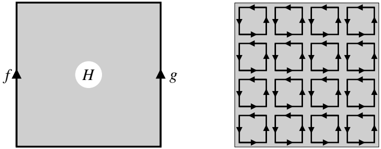


We used:

- Additivity of the winding number for 1-cycles.
- Distant Cycle Lemma.

What happens when $f \simeq_H g$?

Let $M = \min_{s,t} |H(s,t)|$.
Let $\delta > 0$ such that $|s - s'|, |t - t'| < \delta$ implies $|H(s,t) - H(s',t')| < M$.
Divide the square into small squares of side less than δ.



H maps each small square to a quadrilateral with winding number 0.
It follows that $0 = w(g_T, 0) - w(f_T, 0) = w(g, 0) - w(f, 0)$.

We used:

- Additivity of the winding number for 1-cycles.
- Distant Cycle Lemma.

The continuous winding number: normalization

What happens when $f = e^{2\pi i d t}$?

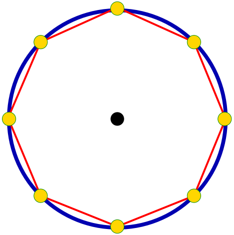
Let

$$T = \left(0, \tfrac{1}{N}, \tfrac{2}{N}, \dots, \tfrac{N-1}{N}, 1\right)$$

where $N > 2|d|$. Then

$$\mathrm{w}(e^{2\pi i d t}, 0) = \mathrm{w}\left(\left[e^{2\pi i d t}\right]_T, 0\right) = \frac{1}{2\pi} \sum_{k=1}^{N} \frac{2\pi d}{N} = d$$

as required.



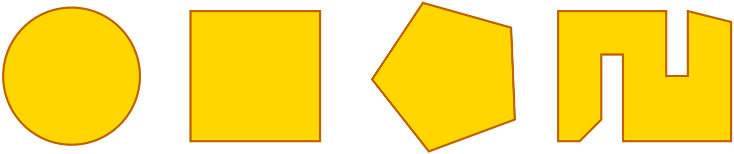
Application: Brouwer's Fixed Point Theorem

Theorem

Every continuous map $p : D^2 \to D^2$ has a fixed point.
(Here D^2 is the closed unit disk in the plane.)

More generally

We may replace D^2 by any space that is topologically equivalent.



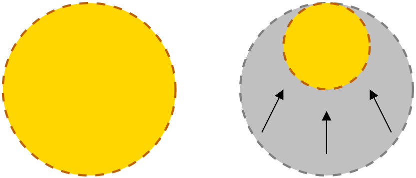
Application: Brouwer's Fixed Point Theorem

Theorem

Every continuous map $p : D^2 \to D^2$ has a fixed point.
(Here D^2 is the closed unit disk in the plane.)

More generally

The theorem fails for the open disk.



Application: Brouwer's Fixed Point Theorem

Theorem

Every continuous map $p : D^2 \to D^2$ has a fixed point.
(Here D^2 is the closed unit disk in the plane.)

More generally

The theorem fails for the annulus.

Application: Brouwer's Fixed Point Theorem

Theorem

Every continuous map $p : D^2 \to D^2$ has a fixed point.
(Here D^2 is the closed unit disk in the plane.)

Proof

Suppose p has no fixed points. Then $q(x) = x - p(x)$ is never zero.

- Let $f(t) = e^{2\pi it}$. $\quad \mathrm{w}(f, 0) = 1$
- Let $J(s,t) = e^{2\pi it} - sp(e^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = q(e^{2\pi it})$. $\quad \mathrm{w}(g, 0) = ?$
- Let $K(s,t) = q(se^{2\pi it})$. $\quad \mathrm{w}(g, 0) = \mathrm{w}(h, 0)$
- Let $h(t) = q(0)$. $\quad \mathrm{w}(h, 0) = 0$

Application: Fundamental Theorem of Algebra

Theorem

Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s,t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$

Application: Fundamental Theorem of Algebra

Theorem

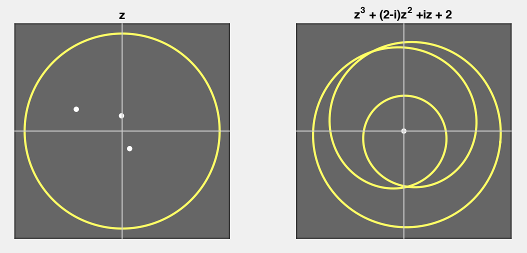
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$



Application: Fundamental Theorem of Algebra

Theorem

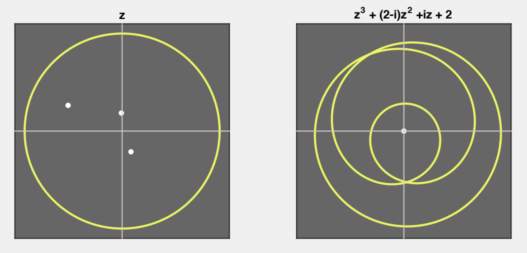
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$



Application: Fundamental Theorem of Algebra

Theorem

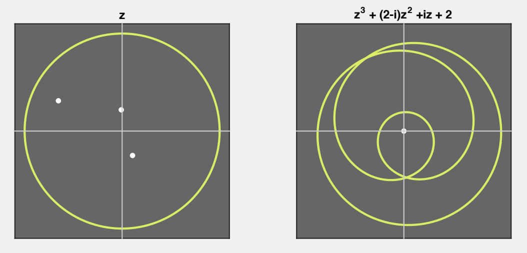
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi i t})$. $\mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi i t})$. $\mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\mathrm{w}(g, 0) = 0$



Application: Fundamental Theorem of Algebra

Theorem

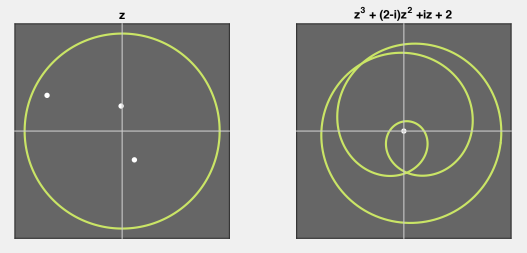
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$



Application: Fundamental Theorem of Algebra

Theorem

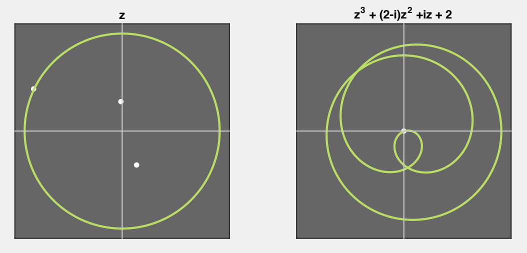
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\mathrm{w}(g, 0) = 0$



Application: Fundamental Theorem of Algebra

Theorem

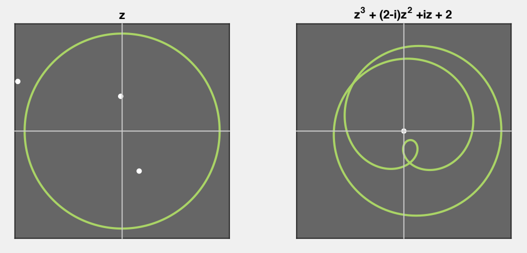
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$



Application: Fundamental Theorem of Algebra

Theorem

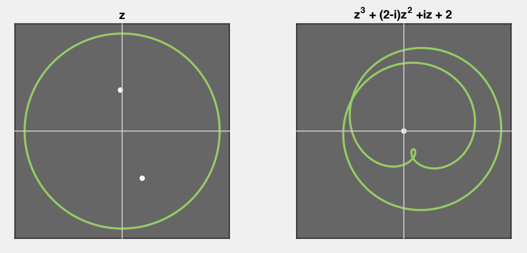
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$



Theorem

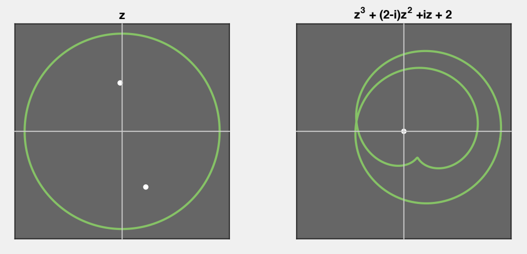
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$



Theorem

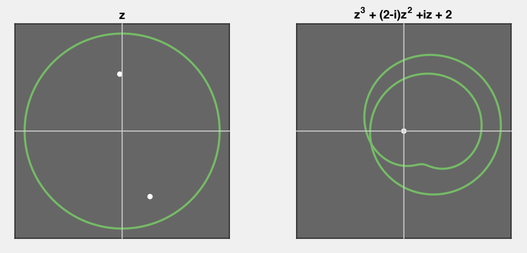
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\mathrm{w}(g, 0) = 0$



Application: Fundamental Theorem of Algebra

Theorem

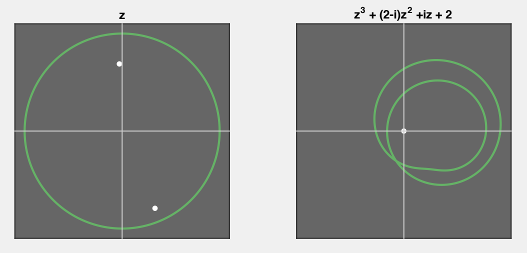
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$



Application: Fundamental Theorem of Algebra

Theorem

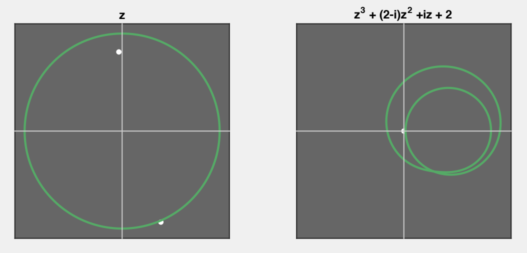
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$



Application: Fundamental Theorem of Algebra

Theorem

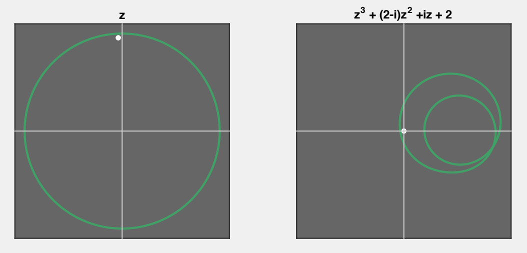
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s,t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$



Application: Fundamental Theorem of Algebra

Theorem

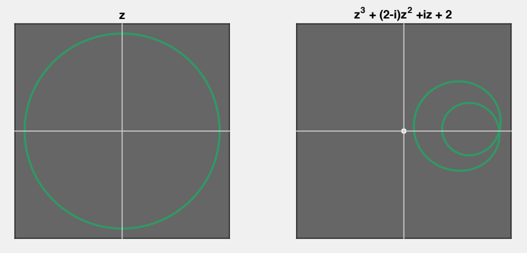
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$



Application: Fundamental Theorem of Algebra

Theorem

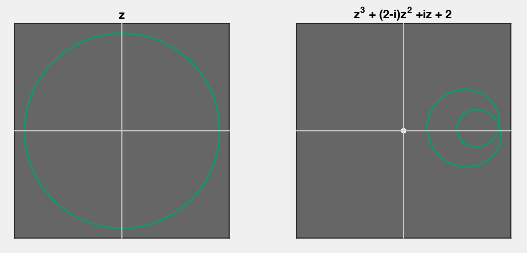
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$



Application: Fundamental Theorem of Algebra

Theorem

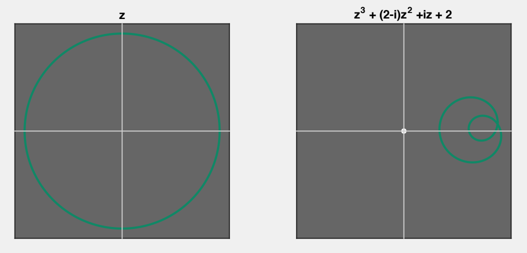
Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad$ $\mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\quad$ $\mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad$ $\mathrm{w}(g, 0) = 0$



Theorem

Every complex polynomial $p(z)$ of degree $d \geq 1$ has at least one complex root.

(By repeated factorization, it therefore has d roots.)

Proof

Suppose p has no roots. Let $R > 0$ be very large.

- Let $f(t) = p(Re^{2\pi it})$. $\quad \mathrm{w}(f, 0) = d$
- Let $H(s, t) = p(sRe^{2\pi it})$. $\quad \mathrm{w}(f, 0) = \mathrm{w}(g, 0)$
- Let $g(t) = p(0)$. $\quad \mathrm{w}(g, 0) = 0$