E4-6𝒢𝒮: Exploring Projective Conics[1]

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Introduction:

In projective geometry there are two types of conics, namely point and line conics. Once again, since the elements of these sets are defined strictly in terms of points and lines, the capabilities of dynamic geometry software greatly enhance the construction and exploration of these sets. The activities below are similar to, and extend, Exercises 1 and 2 in Section 4.6 of Cederberg. Exercises 5 and 6 in that same section are also most efficiently done using dynamic geometry software.

Equipment and Materials Needed:

1. A computer that has access to the program *The Geometer's Sketchpad, Version 3* and a disk on which to save files.

2. Sketches "Hom-cen" and "Hom-axis" constructed in Exploration **E4-5** or the sketches "Pen-lns" and "Pen-pts" used in **E4-5**.

In the first activity below you will find five points of the point conic determined by a projectivity between two pencils of lines. Then you will use a center of homology to find additional points of the point conic. Finally you will observe how the conic changes when you change a line in one of the pencils. Activities 4 and 5 explore line conics.

1. Constructing points of a point conic determined by the projectivity $abc \wedge a'b'c'$:

   (a) Reopen your sketch "Hom-cen.gsp" from Exploration **E4-5** or open the sketch "Pen-lns.gsp" and use the suggestions in activity 5 of that exploration to find the center of homology.

   (b) Hide, but do **not** cut, all the lines and points you used in your construction of H, d' and e'. Do **not** hide H, d' or e'.

   (c) Use **Point at Intersection** under the **Construct** menu to find $A = a \cdot a'$, $B = b \cdot b'$, $C = c \cdot c'$, $D = d \cdot d'$ and $E = e \cdot e'$. Label each of these points.

   (d) Save your sketch as "Pt-Con1.gsp"

2. Finding additional points of the point conic:

   (a) In the same sketch in an open space, use the *Segment* tool (in the *Lines* toolbox) to construct a short segment s.

   (b) Select both $P = a \cdot b$ and segment s and use **Circle by Center and Radius** to construct a circle with center P and a radius congruent to s.

   (c) Use **Point on Object** to construct a point X on this circle. Label X and construct the line $r = PX$.

   (d) Use the center of homology to find r', the line corresponding to r under the projectivity, and construct and label the point $R = r \cdot r'$. Note: You may need to change the location of the point X on your circle. To do so, select point P and choose **Mark Center** in the **Transform** menu. Then use the *Rotate* tool in the *Selection* toolbox to select point X and move it around the circle to a convenient location.

   (e) Change back to the *Select* tool and select the point R. Then choose **Trace Point** under the **Display** menu.

   (f) Deselect R, and then select X.

   (g) Keeping X selected, also select the circle and choose **Animate** under the **Display** menu. In the window that appears, change the animation speed to "slowly."

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[1] Designed to supplement Section 4.6 in *A Course in Modern Geometries, 2nd Ed.*; revised by JNC July 18, 2000

(h) Click on "Animate" and then observe what happens.

(i) Now click the mouse **once only** to see the curve joining the locus points (Clicking a second time will remove the curve.).

(j) Note the shape of the curve and the relation of the points P, P', A, B, C, D and E to this curve.

(k) Create a caption describing both the shape of the locus traced out by the point R and the relation of the locus to the points P, P', A, B, C, D and E.

(l) Save your sketch as "Pt-Con2.gsp"

3. Is a circle a point conic[2]?

(a) In a new sketch, construct a large circle. Then construct 5 points on your circle by using **Point on Object** under the **Construct** menu. Do not use the original point determining your circle as one of the 5 points.. Label these points as P, Q, R, S and T.

(b) Use these 5 points to construct three pairs of corresponding lines a, b, and c through point P and a', b' and c' through point Q and determine a projective correspondence between the lines in the pencils that makes P, Q, R, S and T points of the corresponding point conic (see Section 4.6 of Cederberg). Create a caption giving the projective correspondence.

(c) Find and label H the center of homology determined by this projectivity.

(d) Construct as bold lines the two lines joining the points P and Q to H. What seems to be true about these lines? Extend your caption to indicate your observation.

(e) Follow the procedure in Activity 2 above to generate a locus of points for the point conic generated by this projectivity Where are the points of the locus?

(f) Again, extend your caption to describe your observations.

(g) Save your sketch as "Cir-Pts.gsp"

4. Constructing a line conic determined by the projectivity $ABC \wedge A'B'C'$:

(a) Reopen your sketch "Hom-axis.gsp" from Exploration **E4-5** or open the sketch "Pen-pts.gsp" and use the suggestions in activity 6 of that exploration to find the axis of homology.

(b) Hide, but do **not** cut, all the lines and points you used in your construction of h, D' and E'. Do **not** hide h, D' or E'.

(c) Construct lines AA', BB', CC', DD' and EE'.

(d) Select line EE', change its color to a new color.

(e) With line EE' still selected, choose **Trace Line** under the **Display** menu.

(f) Now deselect EE' and then select both point E and segment p and choose **Animate** under the **Display** menu (Note that the Animate feature requires a *segment*, so p has been constructed as a segment with endpoints off-screen.). During the animation, the point E will disappear off the screen for awhile and then return. Wait until this happens so that you get more lines of the line conic.

(g) Now note the shape that appears to be "enclosed" by the locus of the line EE'. Where are the lines AA', BB', CC', and DD' relative to this shape?

(h) Create a caption describing your observations.

(i) Save your sketch as "Ln-Con1.gsp"

5. Changing the projectivity defining the line conic:

(a) Using the same sketch, select line EE' and turn off **Trace Line**.

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[2]This activity essentially duplicates exercise 1 in Section 4.6 of Cederberg.

(b) Then using the *Select* tool, change the ordering of some of the points defining the projectivity. Be sure you can still see point E.

(c) Redo steps (d) through (f) above.

(d) Change your caption, if necessary, to reflect your current observations.

(e) Save your sketch as "Ln-Con2.gsp"

Report:

Submit a computer folder titled "E4-6GS." This folder should contain the constructions listed below and any other dynamic geometry software sketches you made for the activities in section 4.6 of the text *A Course in Modern Geometries, 2nd. Ed.*[3]

(a) Pt-con1.gsp   (b) Pt-con2.gsp   (c) Cir-Pts.gsp   (d) Ln-con1.gsp   (e) Ln-con2.gsp

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[3]For each of these others, include a comment/caption describing the purpose of the construction.