E4-5$\mathcal{DGS}$: Exploring Projectivities[1]

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Introduction:

The activities below (which essentially duplicate exercises 3, 4, 6, 7, and 9 in section 4.5 of Cederberg) explore the projective transformations known as *projectivities*. These transformations map collinear points to collinear points (or concurrent lines), and dually, concurrent lines to concurrent lines (or collinear points) and are defined entirely in terms of incidence properties. The ability of *Dynamic Geometry Software* to maintain relations of incidence while points and/or lines are dragged about, makes it relatively easy to find constructions where all the appropriate points of intersection are visible. The directions below use the terminology of *Cabri Geometry* and *Geometer's Sketchpad.*[2]

Equipment and Materials Needed:

1. A computer (Macintosh or a PC running Windows) with access to either *Cabri Geometry II* or *Geometer's Sketchpad, Version 3 or 4*, and a disk on which to save files.
2. Three pre-made figures/sketches *Pen-pts*, *Pen-lnpt* and *Pen-lns*.

Exploration Activities:

Construction PP below details a geometric recipe for constructing a projectivity as a sequence of two *perspectivities* between two sets of collinear points. To use this construction, the lines containing the two sets of collinear points must be distinct. The case where the two sets of collinear points lie on the same line, known as a projectivity on a line, is explored in activity 2. Rather than referring to collinear points, the construction refers to "pencils of points". The dual construction then refers to "pencils of lines," i.e., concurrent lines[3].

Construction PP – Images Under a Projectivity Between Distinct Pencils: *Let A, B, C be elements of the pencil with axis p and A', B', C' corresponding elements of the pencil with axis p' ($p' \neq p$). Construct line AA' and choose a point $P \neq A'$ on this line. Let $m \neq p'$ be an arbitrary line through A'. Let $B_1 = BP \cdot m$, $C_1 = CP \cdot m$. Thus, $ABC \overset{P}{\underset{\wedge}{}} A'B_1C_1$. Now let $Q = B_1B' \cdot C_1C'$. Then $A'B_1C_1 \overset{Q}{\underset{\wedge}{}} A'B'C'$ and therefore $ABC \wedge A'B'C'$.*

1. Constructing a projectivity between two distinct pencils of points.
   (a) Open the figure/sketch "Pen-pts."
   (b) Use Construction PP to construct a projectivity that maps A, B, C to A', B', C', respectively ($ABC \wedge A'B'C'$). Be sure to label m, P and Q. Hint: To keep track of which lines belong to which perspectivity, you may want to color the lines in the perspectivity with center P, namely AP, BP and CP, one color and the lines in the perspectivity with center Q, namely QA', QB', and QC', another color.
   (c) Construct D', the image of D, using the same color coding for perspectivities as before.
   (d) With your construction still on the screen, observe what, if anything, happens to D' while you try each of the following:
      - Leaving the points A, B, C, D, A', B' and C' fixed,
         – Drag the point P along the line AA'.

[1] Designed to supplement Section 4.5 in *A Course in Modern Geometries, 2nd Ed.*; revised by JNC July 24, 2002

[2] In particular, the terms "figure" and "comment" are Cabri terms while the terms "sketch" and "caption" are the comparable Sketchpad terms. Also note that the extension used in Cabri file names is ".fig" whereas ".gsp" is used for Sketchpad files.

[3] For further details on perspectivities, projectivities and the notation involved, see Section 4.5 in *A Course in Modern Geometries.*

  - Rotate line m about A'.
- Without moving P, m, and D,
  - Drag, in turn, the points A, B, and C along p.
  - Drag the points A', B', and C' along p'.

(e) Record an on-screen comment/caption explaining your observations, i.e., explain what causes changes in D'. Also explain what this tells you about the information needed to uniquely determine a projectivity.

(f) Save your construction as "Pt-pro1.fig" or "Pt-pro1.gsp"

2. Constructing a projectivity on a pencil of points.

(a) In a new figure/sketch locate four collinear points A, B, C and D. (Be sure that none of these points coincides with the points defining the line.)

(b) Construct the projectivity $ABC \wedge ACB$ using the suggestions below:

- You will need to first use one perspectivity to map the points A, B and C to points A', B' and C' on a second line, and then use Construction PP to map A', B' and C' appropriately. As before, label all points and lines involved, and color code the perspectivities by using a different color for each perspectivity.

(c) Record a comment/caption indicating the perspectivities involved (use the shorthand notation "ABC persp thru P to $A'B'C'$," etc.),

(d) Find and label the image of D.

(e) Record a second comment/caption in which you make a conjecture in answer to the following question:

- Do projectivities preserve the order of collinear points?

(f) Save your construction as "Pt-pro2.fig" or "Pt-pro2.gsp."

3. Constructing a projectivity between pencils of different kinds:

(a) Open the figure/sketch "Pen-lnpt."

(b) Now construct a projectivity $abc \wedge A'B'C'$. Note: You will need to first use one perspectivity to map the lines a, b and c to collinear points A, B and C, and then use Construction PP to map A, B and C appropriately. As before, label all points and lines involved, and color code the perspectivities, by using a different color for each perspectivity.

(c) Record a comment/caption indicating the perspectivities involved (use the notation "abc persp thru p to $a'b'c'$") and explain how you could find the image of a fourth line d concurrent with a, b and c.

(d) Save your construction as "Lnpt-pro.fig" or "Lnpt-pro.gsp"

4. Constructing a projectivity between pencils of lines.

(a) Open the figure/sketch "Pen-lns."

(b) By dualizing Construction PP, carry out the construction of a projectivity that maps a, b, c to a', b', c', respectively. To make your construction easier to decipher, use several colors and use thick lines for the axes of your perspectivities.

(c) Record a comment/caption indicating the perspectivities involved and explain how you could find the image of a fourth line d concurrent with a, b and c.

(d) Save your Construction as "Ln-pro.fig" or "Ln-pro.gsp"

Although the definition of a projectivity requires the construction of a sequence of perspectivities to find the image of lines under a projectivity between pencils of lines, it is more efficient to use the *center of homology* determined by the projectivity. The existence of this point is guaranteed by a theorem in projectivity geometry[4]. The dual of the center of homology is known as the *axis of homology.* Such an axis can be used to find the images under a projectivity between pencils of points. The activities below lead to the construction of projective images using a center and an axis of homology. These particular methods will be used in Exploration **E4-6** to simplify the constuction of conics.

Construction CH – Finding projective images via a center of homology: H, the center of homology determined by the projectivity $abc \wedge a'b'c'$ is the point of intersection of the line joining $a' \cdot b$ with $a \cdot b'$ and the line joining $a' \cdot c$ with $a \cdot c'$. To use H to find d', the image of line d under this projectivity, first find d_1 the line joining $a' \cdot d$ with H. Then d' is the line joining the point $a \cdot d_1$ with $a' \cdot b'$.

5. Constructing a projectivity between pencils of lines using the center of homology.
   (a) Open the figure/sketch "Pen-lns"
   (b) To find H, the center of homology determined by the projectivity $abc \wedge a'b'c'$:
      i. Construct the points $a' \cdot b$, $a \cdot b'$, $a' \cdot c$, $a \cdot c'$.
      ii. Then construct the line joining $a' \cdot b$ with $a \cdot b'$ and the line joining $a' \cdot c$ with $a \cdot c'$. (You may want to use a new color for these lines.)
      iii. Construct and label H, the point of intersection of these two lines.
   (c) Using the center H, find d' and e', the images of d and e under this projectivity.
   (d) Record a comment/caption giving a brief explanation of the procedure you used to construct lines d' and e'.
   (e) Save your construction as "Hom-cen.fig" or "Hom-cen.gsp"
6. Constructing a projectivity between pencils of points using the axis of homology.
   (a) Open the figure/sketch "Pen-pts"
   (b) Find and label h, the axis of homology determined by the projectivity $ABC \wedge A'B'C'$. (Do *not* construct the projectivity.) Make h a different color.
   (c) Use your axis h to find D' and E', the images of D and E under this same projectivity. Label these points.
   (d) Record a comment/caption giving a brief explanation of the procedure you used to find the axis h and the way you used h to construct points D' and E'.
   (e) Save your construction as "Hom-axis.fig" or "Hom-axis.gsp"

Report:

Submit a computer folder titled "E4-5CG" or "E4-5GS" (depending on whether you used *Cabri* or *Sketchpad*). This folder should contain the constructions listed below and any other dynamic geometry software figures/sketches you made for the activities in section 4.5 of the text *A Course in Modern Geometries, 2nd. Ed.*[5] For each, be sure to use the appropriate extension, i.e., ".gsp" for *Sketchpad* sketches and ".fig" for *Cabri* figures.

(a) Pt-pro1 (b) Pt-pro2 (c) Lnpt-pro (d) Ln-pro (e) Hom-cen (f) Hom-axis

[4] Use the dual of Theorem 4.11 in section 4.5 of Cederberg.

[5] For each of these others, include a comment/caption describing the purpose of the construction.