E4-4$\mathcal{DGS}$: Exploring Harmonic Sets[1]

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Introduction:

The activities below (which essentially duplicate exercises 1, 2, 5 and 6 in Cederberg, section 4.4) explore the relation known as a *harmonic set*. This relation, like others in projective geometry, is defined entirely in terms of incidence properties. Thus the elements involved in the relation can all be constructed entirely with lines and points. Since pairs of lines always intersect in projective geometry, it is essential to be able to locate all points of intersection in these constructions. The ability of *Dynamic Geometry Software* to maintain relations of incidence while points and/or lines are dragged about, makes it relatively easy to find constructions where all the appropriate points of intersection are visible. The directions below use the terminology of *Cabri Geometry* and *Geometer's Sketchpad.*[2]

Equipment and Materials Needed:

A computer (Macintosh or a PC running Windows) with access to either *Cabri Geometry II* or *Geometer's Sketchpad, Version 3 or 4*, and a disk on which to save files.

Exploration Activities:

Before beginning, you should refer to the definition of a harmonic set found in section 4.4 of Cederberg, *A Course in Modern Geometries.* Once three collinear points A, B, C are given, a fourth point D, called the *harmonic conjugate or C with respect to A and B*, written $H(AB, CD)$, can be found using Construction HP described below. You will need to use this construction, or its dual, in the following activities. In each case, be sure to verify that the points A, B, C and D (or lines a, b, c and d) satisfy the definition of a harmonic set.

Construction HP – the Fourth Point of a Harmonic Set: *Let E be an arbitrary point not on AB and let m be a line through B that is distinct from AB and not incident with E. Let $m \cdot AE = F$, $m \cdot CE = G$, and let $AG \cdot EB = H$. Then $D = AB \cdot FH$.*

1. Open a new figure/sketch and in it construct a line. On this line, construct three new points A, B and C, with B lying "between[3]" A and C. (**Important Note:** Be sure that none of the points A, B and C coincides with either of the two original points defining your line.)
   (a) Now use Construction HP to locate D, the harmonic conjugate of C with respect to A and B. Be sure to label m and each of the points E, F, G and H.
   (b) With your construction still on the screen, leave the points A, B and C fixed and observe what happens to D while you try each of the following:
      - Drag the point E to various locations. (Note: E cannot be incident with line AB.)
      - Rotate m about point B. (Note: m cannot coincide with line AB.)
   (c) Record an on-screen comment/caption summarizing your observations and what this tells you about the uniqueness of the harmonic conjugate.
   (d) Save your construction as "Har-pt1.fig" or "Har-pt1.gsp"
2. With your construction still on the screen, eliminate your comment/caption.
   (a) Observe what happens to the positions of C and D relative to A and B, as you leave E and m invariant and move, in turn, points A, B and C along the line AB.

[1]Designed to supplement Section 4.4 in *A Course in Modern Geometries, 2nd Ed.*; revised by JNC July 24, 2002

[2]In particular, the terms "figure" and "comment" are Cabri terms while the terms "sketch" and "caption" are the comparable Sketchpad terms. Also note that the extension used in Cabri file names is ".fig" whereas ".gsp" is used for Sketchpad files.

[3]Since this term is not formally defined in projective geometry, use the standard Euclidean interpretation of the term.

(b) In particular, note what happens when C is located "half-way" between A and B.

(c) Record an on-screen comment/caption describing your observations (Be sure to explain what happens in the case where C appears to be "half-way" between A and B.).

(d) Save your construction as "Har-pt2.fig" or "Har-pt2.gsp"

3. In a new figure/sketch, locate a point P and construct three lines a, b and c incident with P.

   (a) Dualize Construction HP to obtain d, the harmonic conjugate of c with respect to a and b. To make your construction easier to decipher, make your construction lines using a second color different from the color of lines a, b and c. Then use a third color for d. Also, label all the lines and points used in your construction.

   (b) Record a comment/caption summarizing your construction.

   (c) Save your construction as "Har-ln.fig" or "Har-ln.gsp"

4. (Optional) In a new figure/sketch[4], construct a segment of length 6 in. Label the left endpoint "O" and the right endpoint "C."

   (a) Construct point G two-thirds of the distance from O to C and point E two-fifths of the way from G to C, i.e, make $d(O,G) = 4$ in, $d(G,E) = .8$ in, $d(E,C) = 1.2$ in.

   (b) Use Construction HP to obtain a quadrangle verifying that $H(OE, CG)$.

   (c) Save your construction as "Har-str.fig" or "Har-str.gsp"

   Note: This exercise shows why the term "harmonic" is used to refer to such a set of four points, since if the segment OC represents a stretched string tuned to the note C, the same string stopped at E or G will play the other notes of the major triad.

Report:

Submit a computer folder titled "E4-4CG" or "E4-4GS" (depending on whether you used *Cabri* or *Sketchpad*). This folder should contain the constructions listed below and any other dynamic geometry software figures/sketches you made for the activities in section 4.4 of the text *A Course in Modern Geometries, 2nd. Ed.*[5]. For each, be sure to use the appropriate extension, i.e., ".gsp" for *Sketchpad* sketches and ".fig" for *Cabri* figures.

(a) Har-pt1 (b) Har-pt2 (c) Har-ln (d) Har-str (optional)

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[4]This exercise is adapted from Coxeter (1987, *Projective Geometry*, p. 23).

[5]For each of these others, include a caption describing the purpose of the construction.