

E2-GS: Exploring the Hyperbolic Plane With Sketchpad¹

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Introduction:

The plane non-Euclidean geometry known as hyperbolic geometry can be modeled in the interior of a Euclidean circle. The model known as the Poincaré model uses the interpretations given in the table below.

Poincaré Model	
Hyperbolic Term	Euclidean Interpretation
(Ordinary) Point	Point interior to a given Euclidean circle \mathcal{C}
Ideal Point	Point on the circumference of \mathcal{C} Note: Ideal points are not actual points in hyperbolic geometry.
Line	Portion interior to \mathcal{C} of diameter of \mathcal{C} or of a circle orthogonal to \mathcal{C}
On (a point is on a line)	The Euclidean point lies on the cord

The following activities ask you to explore the geometry modeled by the Poincaré disk model and develop conjectures based on your explorations.

Equipment and Materials Needed:

1. A computer (either Macintosh or PC running Windows) that has access to the program *The Geometer's Sketchpad (GS)*, Version 3 or 4 and a disk on which to keep a copy of your work.
2. The Poincaré disk tools:
 - For *GS 4*: These come in the premade sketch *Poincare Disk.gsp* that comes with your copy of *Sketchpad* and is located in the *Sketchpad* Program folder:
Samples/Sketches/Investigations
 - For *GS 3*: You need to download and install the tools for the Poincaré disk model and the sketch PoinStrt.gsp. These are available at the following web site:
<http://mathforum.org/sketchpad/gsp/gallery/poincare/poincare.html>
These tools should be loaded in *Sketchpad's* Script Tool Box. Do this by opening the **Display** menu, choosing first *Preferences* and then *More*. In the **Dialog Box** that appears, press the *Set* button for the *Script Tool Folder* and then open the appropriate folder in the list.

Exploration Activities:

The following activities ask you to use several special *Sketchpad* tools to explore the geometry modeled by the Poincaré disk model and develop conjectures based on your explorations. These tools will be located in the Custom/Script Tool Box. To use these, you first select the Tool to be used and then the necessary objects.

¹Designed to supplement Chapter 2 (in particular, Section 2.3) in *A Course in Modern Geometries, 2nd Ed.*; revised by JNC July 23, 2002

2.1.1 Points and Lines:

1. To set up the circle \mathcal{C} described above:
GS 4: Open the sketch *Poincare Disk.gsp* found in the *Sketchpad* program folder: *Samples/Sketches/Investigations*.²
GS 3: Open the sketch *PoinStrt.gsp*.
2. Construct and label a pair of points A and B inside the disk. Then use the following procedure to measure the hyperbolic distance, $d(A, B)$. Go to the Tool Box and click and hold on the Custom/Script Tools icon (the last icon). Select the *Hyperbolic Distance* tool. Then match points A and B by clicking on the two points in succession. The distance will then appear on your screen. If necessary, relabel as “ $AB =$ ”
3. With both A and B selected, move point A (B should come along) to different locations inside the model. Throughout this process A and B should appear to be the same distance apart, but note what happens to the hyperbolic distance, $d(A, B)$.
4. Using the *Hyperbolic Line* tool,³ construct the line determined by A and B . Again, note what happens as you move A and B to different locations.
5. Construct and label several more pairs of points and the lines they determine. Also find and label the distances between the points in your pairs.
6. **Summary:** Using observations from these constructions, use the *text tool* to open a caption box and write in it a *paragraph* summary describing the apparent properties of points and lines in the Poincaré disk model of the hyperbolic plane. In particular, your summary should include answers to the following:
 - (a) Describe the appearance of the lines. Do they always look the same? How many points determine a line?
 - (b) How does the hyperbolic distance between pairs of points change as the points get closer to the edge of the bounding circle?
 - (c) If ℓ and m are non-intersecting lines and m and n are non-intersecting lines, what, if anything, can you say about lines ℓ and n ?
 - (d) Does this geometry satisfy Playfair’s axiom? If not, can Playfair’s axiom be modified so that it becomes a true statement in this model? If so, how?
7. Close and save your construction and summary as “Points-Lines.gsp” or “Pts-Lns.gsp”

2.1.2 Triangles:

1. Again open the sketch *Poincare Disk* (or *GS 3: PoinStrt.gsp*) to obtain a new model.
2. In this model, construct 3 noncollinear points A , B and C . Then use the *Hyperbolic Segment* tool to construct the sides of triangle $\triangle ABC$.
3. Use the *Hyperbolic Angle* tool to measure the angles in $\triangle ABC$.
4. Under the **Measure** menu, choose the *Calculate* option. By highlighting each of the three angle measures and inserting addition signs, calculate the sum of the angle measures, i.e., the angle sum, of $\triangle ABC$.
5. Observe what happens to this sum as you lengthen the sides and change the position of $\triangle ABC$.

²To move or resize the circle that appears, click on the *Disk Control* button to display the points that determine the center and radius of the circle. Then select and drag the center point to relocate the circle and the circumference point to resize the circle.

³This and all other hyperbolic tools should be listed in the Custom Tool Box.

6. **Summary:** Using observations from these constructions, use the *text tool* to open a caption box and write in it a *paragraph* summary describing the apparent properties of triangles in the disk model of the hyperbolic plane. In particular, your summary should include answers to the following:
 - (a) How does the angle sum of a triangle in this model compare to the angle sum of a triangle in the Euclidean plane? Is this sum the same for all triangles?
 - (b) What will be true of the angle sum of a four-sided figure? A five-sided figure?
7. Close and save your construction and summary as “Triangles.gsp” or “Tri.gsp”

2.1.3 Perpendiculars and Non-intersecting Lines:

1. Again open the sketch *Poincare Disk* (or *GS 3: PoinStrt.gsp*) to obtain a new model.
2. In this model, construct a line AB and a point P not on AB . To keep track of everything, be sure to label all your points and lines as indicated.
3. Now select the *Hyperbolic Perpendicular* (*GS 3: PerpPt1*) tool and click first on P and then on points A and B to construct a line through P that is perpendicular to AB . Construct and label the point of intersection of these two lines as C .
4. Measure one of the angles formed by AB and PC using the *Hyperbolic Angle* tool. If necessary, relabel the angle measure appropriately, e.g., as “ $m\angle ACP =$ ” or “ $ACP =$ ”
5. Construct line ℓ perpendicular to PC through P . (Since point P is on the line, you will need to use a method of construction differing from that used in part 3 above.)
 In *GS4*: You can construct this line by selecting the *Hyperbolic Circle by CP* tool and then clicking in order on points P and C to first construct a circle with center P and radius PC . Find C' , the point of intersection of this circle with line PC . Finally use the *Hyperbolic Perpendicular Bisector* tool to construct the perpendicular bisector of segment CC' .
 In *GS3*: You can use the *PerpPt2* tool.
6. Using *Point on Object* under the **Construct** menu, construct a second point Q on line ℓ .
7. Find (and label) $m\angle CPQ$.
8. On line AB construct and label a point D distinct from A , B and C . Using the procedure in part 5 above, construct a line m through D perpendicular to AB .
9. Move point D to different locations on line AB . When m intersects PQ , construct the point of intersection and label it as R . Also move point Q along PQ until R lies between P and Q .
10. Find (and label) $m\angle CDR$ and $m\angle PRD$.
11. Find (and label) the distances $d(P, C)$ and $d(D, R)$.
12. Now, drag point D along line AB and observe what happens to your distance and angle measurements.⁴
13. Finally, use the *Hyperbolic Line* tool to construct a new line ρ through P distinct from the lines PC and PQ . Change its color so that it is easily distinguishable from your other lines. Now use the pointer to rotate ρ around the point P by dragging the second point on ρ . Watch what happens to the “intersection” of ρ and AB as you do so.

⁴In *GS3*: To maintain an on-screen record of the angle and distance measures even as you move D past C , you will need to do the following: Drag point D to the other side of C and again find and rename the following: $m\angle CDR$, $m\angle PRD$, and $d(DR)$.

14. **Summary:** Using observations from these constructions, use the *text tool* to open a caption box and write in it a *paragraph* summary describing the apparent properties of perpendicular and non-intersecting lines in the disk model of the hyperbolic plane. In particular, your summary should include answers to the following:
- Can a perpendicular always be drawn to a given line through a given point? Does it matter if the point is on the given line?
 - How many common perpendiculars exist between 2 non-intersecting lines?
 - Are two non-intersecting lines equidistant?
 - If P is a point not on line AB , how many lines can be drawn through P that do not intersect AB ?
 - When you rotate line ρ about the point P away from line PC , is there a last position in this rotation (in either the clockwise or counterclockwise directions) where ρ intersects AB , i.e., that has an H-point in common with AB ? Is there a “first position” in this same rotation where ρ does not intersect AB ?
15. Close and save your construction and summary as “Perpend-Nonintersect.gsp” or “Per-noin.gsp”

2.1.4 Special Non-intersecting Lines:

As you may have discovered in your explorations in the Poincaré model, there are some special lines in *hyperbolic* geometry.

- **Sensed Parallel Lines:** Given a line AB and a point P not on AB , there are two lines through P that are “first” lines in the rotation from the perpendicular to AB that do not intersect AB , one line on each side of the perpendicular. These “first” lines are said to be *sensed parallel* to AB .

Sensed parallels play an important role in the theory of hyperbolic geometry. The following exercises guide you through a construction using only points in the hyperbolic plane to generate these “first” lines. This construction is detailed and justified on pages 93-96 of Wolfe’s *Introduction to Non-Euclidean Geometry* (1945).

- First, read through the entire set of directions below. Then carry out the construction.⁵ Note: You must use the special hyperbolic tools to construct lines and measure angles and distances. Use the ordinary *Sketchpad* tools to construct points.
- **To Construct a Sensed Parallel to line AB through point P :**
Construct a line AB and a point P not on AB . Now draw a line through P that is perpendicular to AB . Construct the point of intersection of these two lines and label it C . At P draw the line ℓ perpendicular to line PC . Construct a second point Q on ℓ . Also construct D a fourth point on line AB relatively near point C . Then construct the line m perpendicular to line PQ from D . Let R be the point of intersection of lines PQ and m . Use the *Hyperbolic Circle by CR* (*GS3: Centrrad*) tool to construct a circle with center P and radius equal to $d(C, D)$. Construct the point of intersection of this circle with segment DR and label this intersection point as S . Finally construct line PS . This will be one of the sensed parallels to AB through P .
- With the above construction still open, generate the other sensed parallel to AB through the point P . Change the color of your 2 sensed parallels. Then use the *text tool* to open a caption box and write in it a title for your sketch indicating that the [new color] lines are the sensed parallels to line AB through P .
- Close and save your construction and summary as “S-Parallels.gsp” or S-Par.gsp”
- (Optional) Euclidean Sensed Parallels?**

⁵Note: The initial steps in this construction (through the construction of Q) are the same as those used in 2.1.3.

- (a) Carry out the same sensed parallel construction in the Euclidean plane after reading the directions below:
- You should *not* open the *Poincare Disk.gsp* (*PoinStrt.gsp*) sketch and you should use *only* the Euclidean tools and operations found in the regular *Sketchpad* Menus and Toolbox.
 - To construct the circle with center P and radius equal to $d(C, D)$ you will need to first construct the segment CD (on top of the line). Then select both that segment and the point P and finally choose the operation *Circle By Center and Radius* in the **Construct** menu.
- (b) Record a comment giving a short description of the result of your construction of a “sensed parallel” in the Euclidean plane. In particular, describe how points S and R appear to be related.
- (c) Close and save your construction and summary as “E-Sensed-Parallel.gsp” or as “E-S-Par.gsp”

Report:

Submit a computer folder entitled “E2-GS.” This folder should contain:

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|------------------------------|--------------------------------------|
| (a) Points-Lines.gsp | (d) S-Parallels.gsp |
| (b) Triangles.gsp | (e) E-Sensed-Parallel.gsp (optional) |
| (c) Perpend-Nonintersect.gsp | |