

## E2-CG: Exploring the Hyperbolic Plane with Cabri<sup>1</sup>

### Introduction:

The plane non-Euclidean geometry known as hyperbolic geometry can be modeled in the interior of a Euclidean circle. The model known as the Poincaré model uses the interpretations given below.

Poincaré Model	
Hyperbolic Term	Euclidean Interpretation
(Ordinary) Point	Point interior to a given Euclidean circle $\mathcal{C}$ (the <i>horizon</i> in <i>Cabri</i> ).
Ideal Point	Point on the circumference of $\mathcal{C}$ Note: Ideal points are not actual points in hyperbolic geometry.
Line	Portion interior to $\mathcal{C}$ of diameter of $\mathcal{C}$ or of a circle orthogonal to $\mathcal{C}$
On (a point is on a line)	The Euclidean point lies on the cord

The following activities ask you to explore the geometry modeled by the Poincaré disk model and develop conjectures based on your explorations. These activities make use of the file *HypMenUS.men* containing special hyperbolic menus developed by Jean-Marie Laborde, one of the co-developers of *Cabri Geometry*. Mac and Windows versions of this file are available on the St. Olaf server or from the ftp site:

<ftp://ftp.imag.fr/pub/CABRI/Menus-folder/>.

### Setting Up Cabri's Hyperbolic Plane:

- **Loading the Hyperbolic Tools:** Open Cabri Geometry II+ as usual. Then open the hyperbolic menu file *HypMenUs.men* located in the Class Materials folder on the L(Classes) server. This will load hyperbolic menus that appear in the following 6 hyperbolic toolboxes

**H-Toolboxes:** Points (H-Points), H-Lines, H-Curves, H-Construct, H-Transform, H-Measure

These appear, along with the Macro toolbox between the regular Cabri Pointer and Display toolboxes. Further to the right, are the Draw toolbox and toolboxes containing most of the other Euclidean Cabri tools (with some toolboxes combined). Finally, select the *Help* option in the **Help** menu so that tool-use directions will appear at the bottom of the screen.

- **Generating the Hyperbolic Horizon:** Use the *Horizon* tool found in the **H-Lines** toolbox to construct a circle to serve as the hyperbolic horizon  $\mathcal{C}$  described in the table above. Before you continue, you may want to (1) enlarge both the window and your horizon boundary, and (2) label your horizon. However, once you begin your constructions, the horizon **must** remain unaltered.

### Exploration Activities:

#### 2.1.1 Points and Lines:

1. With the Hyperbolic Horizon on screen, use the *Point* tool to construct and label a pair of points  $A$  and  $B$  inside your horizon<sup>2</sup>. Use the *H-Line* tool to construct the line  $AB$ . Note: For this and many other hyperbolic constructions, you will need to select both the objects normally required by the comparable Euclidean construction and the horizon circle.

<sup>1</sup>Designed to supplement Chapter 2 (in particular, Section 2.3) in *A Course in Modern Geometries, 2nd Ed.*; revised by JNC February 5, 2007. Source File: My Documents\Text.dir\Web-exp.dir\e2cg+07.tex.

<sup>2</sup>Recall that you can label a newly constructed point by typing the label immediately after its construction.

2. Use the *H-distance* tool to record the hyperbolic distance,  $d_H(A, B)$ ; be sure to label the distance using the *Text* tool.
3. Use the *Pointer* to drag point  $A$  and then  $B$  to different positions within your horizon.
4. Construct and label several more pairs of points and the lines they determine. Also record and label the distances between the points in your pairs.
5. **Summary:** Using observations from these constructions, use the *Text* tool to record a *paragraph* summary describing the apparent properties of points and lines in the Poincaré disk model of the hyperbolic plane. In particular, your summary should include answers to the following:
  - (a) What do lines “look” like? Do they always look the same? How many points determine a line?
  - (b) How does the hyperbolic distance between pairs of points appear to change as the points get closer to the edge of the horizon circle?
  - (c) If  $\ell$  and  $m$  are non-intersecting lines and  $m$  and  $n$  are non-intersecting lines, what, if anything, can you say about lines  $\ell$  and  $n$ ?
  - (d) Does this geometry satisfy Playfair’s axiom? If not, can Playfair’s axiom be modified so that it becomes a true statement in this model? If so, how?
6. Close and save your construction and summary as “Pts-Lns.fig”

### 2.1.2 Triangles:

1. In a new drawing window, construct another Poincaré model by setting up a horizon.
2. In this model, construct 3 noncollinear points  $A$ ,  $B$  and  $C$ . Then use the *Segment* tool in the **H-Construct** toolbox to construct the sides of triangle  $\triangle ABC$ .
3. Use the *H-Angle* tool to find  $m_H(\angle CAB)$ . Label the measurement and move it to a point outside of your model. Similarly find, label and locate the other two angle measures. Note: Angle measures will be given in radians.
4. Use the *Calculate* tool (in the **H-Measure**) toolbox to calculate the angle sum of  $\triangle ABC$  by selecting sequentially each of the three angle measures, inserting addition signs as needed.
5. Click on the “=” sign to get your result. Register your result on-screen by clicking on the result and then moving your cursor to the desired position. Use *Text* to relabel your result as “Angle Sum.”
6. Use the *H-Area (Triangle)* tool to find the hyperbolic area of  $\triangle ABC$ . Then label as “Triangle Area” and relocate this measure also.
7. Finally, use the *Calculate* tool to calculate the difference  $\pi - \text{angle sum}$ . Label this result as “Angle Deficit.”

### 8. Tabulating the Data:

To analyze your triangle data more carefully, it is helpful to tabulate the data for several different triangles. The following procedure will create a data table.

- (a) Select *Tabulate* from the **H-Measure** toolbox. Then click in a spot where you want the upper left-hand corner of your table to appear. When the table appears, grab its lower right-hand corner and drag down and to the right until you have 6 columns and at least 10 rows.

- (b) To put your data in the table, decide first on the order in which you want your data to appear across the table, e.g. *Angle Measures*, followed by *Angle Sum*, *Triangle Area* and *Angle Deficit*. Then click in turn on each numerical value in your determined ordering. The labels for the data should appear in the first row of your table, while the data itself appears in the second row<sup>3</sup>.
  - (c) To record data for more triangles, select first the *Tabulate* tool and then the *Pointer*. Use the pointer to slowly drag one vertex at a time, pressing the *Tab* key on your keyboard periodically as you do so.
9. **Summary:** Using observations from these constructions, use the *Text* tool to record a *paragraph* summary describing the apparent properties of triangles in the disk model of the hyperbolic plane. In particular, your summary should include answers to the following:
- (a) How does the angle sum of a triangle in this model compare to the angle sum of a triangle in the Euclidean plane? Is this sum the same for all triangles?
  - (b) What will be true of the angle sum of a four-sided figure? A five-sided figure?
  - (c) What, if anything, happens to the angle sum of your triangle as the area increases? as the area decreases?
  - (d) What, if any, relation is there between the angle sum and area of your triangles?
10. Close and save your construction and summary as “Tri.fig”

### 2.1.3 Perpendiculars and Non-intersecting Lines:

NOTE: In this and the following sections you will continue to explore the Poincaré model of the hyperbolic plane. In this model, “lines” are “h-lines” constructed using the *H-Line* tool.

1. In a new Poincaré model, construct a line  $AB$  and a point  $P$  not on  $AB$ . Now use the *Perpendicular* tool in the **H-Construct** toolbox to construct a line through  $P$  that is perpendicular to  $AB$ . Label the point of intersection of these two lines as  $C$ . Use the *Mark Angle* tool in the third toolbox from the right to mark  $\angle ACP$ <sup>4</sup>.
2. Construct and label another line  $m$  perpendicular to  $CP$  through  $P$ . Then locate and label a second point  $Q$  on  $m$  and mark  $\angle CPQ$ .
3. On line  $AB$  locate and label a point  $D$  distinct from  $A$ ,  $B$  and  $C$ . Then construct a line  $n$  through  $D$  perpendicular to  $AB$ . Also construct a second point  $E$  on  $n$  and mark  $\angle ADE$ .
4. Move point  $D$  to different locations on line  $AB$ . When  $n$  intersects  $PQ$ , construct the point of intersection and label it as  $R$ . Also move point  $Q$  along  $PQ$  until  $R$  lies between  $P$  and  $Q$ .
5. Find and label  $m_H(\angle CDR)$ ,  $m_H(\angle PRD)$ ,  $d_H(C, P)$  and  $d_H(D, R)$ .
6. Now move point  $D$  along line  $AB$  while observing your distance and angle measurements. Note: If point  $R$  disappears and  $d_H(D, R)$  becomes “non-existent,” you will need to relabel point  $R$  and set up a second measure for  $d_H(D, R)$ .
7. Finally, construct a new line  $o$  through  $P$  and a new point  $S$  so that line  $o$  is distinct from  $PC$  and  $PQ$ . Change its color so that it is easily distinguishable from your other lines. Now use your pointer to “grab” point  $S$  and rotate line  $o$  around  $P$  until it coincides with  $PC$ . Then rotate  $o$  away from  $PC$ , watching what happens to the “intersection” of lines  $o$  and  $AB$  as you do so.

<sup>3</sup>To change the width of a column, click the *Pointer* on the table to activate the edit window; then point to a column divider in the first row, wait for the column width cursor to appear, and drag it. To delete a data column (row), click on the first row of the column (the first column in the row) to select the column (row) and press the *delete* key.

<sup>4</sup>The *Mark Angle* tool is marking the Euclidean angles, but here we are using it to keep track of the hyperbolic angles.

8. **Summary:** Using observations from these constructions, use the *Text* tool to record a *paragraph* summary describing the apparent properties of perpendicular and non-intersecting lines in the Poincaré disk model of the hyperbolic plane. In particular, your summary should include answers to the following:
- (a) Can a perpendicular always be drawn to a given line through a given point? Does it matter if the point is on the given line?
  - (b) What is the maximum number of right angles in a quadrilateral?
  - (c) How many common perpendiculars exist between 2 non-intersecting lines?
  - (d) Are two non-intersecting lines equidistant?
  - (e) If  $P$  is a point not on line  $AB$ , how many lines can be drawn through  $P$  that do not intersect  $AB$ ?
9. Close and save your construction and summary as “Per-noin.fig”

#### 2.1.4 Special Non-intersecting Lines:

As you may have discovered in your explorations in the Poincaré model, there are some special lines in *hyperbolic* geometry.

**Sensed Parallel Lines:** Given a line  $AB$  and a point  $P$  not on  $AB$ , there are two lines through  $P$  that are “first” lines in the rotation from the perpendicular to  $AB$  that do not intersect  $AB$ , one line on each side of the perpendicular. These “first” lines are said to be *sensed parallel* to  $AB$ .

Sensed parallels play an important role in the theory of hyperbolic geometry. The following exercises guide you through a construction using only points in the hyperbolic plane to generate these “first” lines. This construction is detailed and justified on pages 93-96 of Wolfe’s *Introduction to Non-Euclidean Geometry* (1945).

1. **Constructing a line through  $P$  sensed parallel to line  $AB$ :**<sup>5</sup>
  - (a) In a new Poincaré model, construct a line  $AB$  and a point  $P$  not on  $AB$ . Now draw a line through  $P$  perpendicular to  $AB$ . Construct the point of intersection of these two lines and label it  $C$ . At  $P$  draw the line  $\ell$  perpendicular to line  $PC$ . Construct a second point  $Q$  on  $\ell$ . Also construct  $D$  a fourth point on line  $AB$  relatively near point  $C$ . Then construct the line  $m$  perpendicular to line  $PQ$  from  $D$ . Let  $R$  be the point of intersection of lines  $PQ$  and  $m$ . Use the *H-Compass* tool in the **H-Construct** toolbox to construct a circle with radius equal to  $d_H(C, D)$  and center  $P$ .
  - (b) Find one of the points of intersection of this circle with segment  $DR$  and label this intersection point as  $S$ . Construct line  $PS$ . This will be one of the sensed parallels to  $AB$  through  $P$ .
  - (c) Now use the *Reflection* tool in the **H-Transform** toolbox to reflect point  $S$  across line  $PC$  to obtain point  $S'$ . Then construct line  $PS'$ . This is the second sensed parallel to  $AB$  through point  $P$ . Change the color of your 2 sensed parallels and use the *Text* tool to title your figure indicating that the [new color] lines are the sensed parallels to line  $AB$  through  $P$ .
2. **Summary** Using observations from this construction, use the *Text* tool to record a *paragraph* summary describing the apparent properties of sensed parallel lines in the Poincaré disk model of the hyperbolic plane. In particular, your summary should include answers to the following:
  - (a) Do h-lines  $PS$  and  $PS'$  intersect line  $AB$  within the Poincaré disk model of the hyperbolic plane? Explain.
  - (b) If h-lines  $PS$  and  $PS'$  were extended beyond the interior of the “hyperbolic horizon,” would they intersect line  $AB$  in the Euclidean plane? If so, where?

---

<sup>5</sup>Note: The initial steps in this construction (through the construction of  $Q$ ) are the same as those used in 2.1.3.

- (c) Assume that a h-line  $\rho$  intersects  $AB$  at point  $X$ . If a line  $PQ$  is drawn from a point  $P$  on  $\rho$  perpendicular to  $AB$  at point  $Q$  and  $\rho$  is rotated away from  $PQ$  so that  $X$  moves toward the horizon, is there a “last position” for  $\rho$  where  $\rho$  still intersects line  $AB$ ? Why?

3. Save your completed construction as “S-Par.fig”

**Optional Activities:** The activities below explore sensed parallelism in the Euclidean plane.

To ensure that you carry out constructions in the Euclidean rather than the hyperbolic plane, close *Cabri Geometry* and then reopen it without installing the special hyperbolic menus.

4. **Constructing an  $e$ -line through  $P$  sensed parallel to line  $AB$ :**

- (a) In the Euclidean plane carry out the same construction described in activities 1(a)-(c) above.  
 (b) Use the *Parallel* tool in the **Check Property** toolbox to check whether or not your line  $PS$  is parallel to  $AB$ . Edit the property comment so that it names the lines  $PS$  and  $AB$ . What does this tell you about the construction?

5. **Summary** Using observations from your construction of a sensed parallel in the Euclidean plane, summarize the apparent properties of sensed parallels in the Euclidean plane. In particular, your summary should describe how points  $S$  and  $R$  appear to be related. To check your conjecture you can use the *Member* tool in the **Check Property** toolbox to find out if  $R$  is on your compass circle.

6. Close and save your construction and summary as “E-S-Par.fig.”

## Report:

Your folder “E2-\*\*\*\*\*” should contain the files below. Submit your work by placing a copy of this folder in the Class Drop Box.

- |                   |                            |
|-------------------|----------------------------|
| (a) Pts-Lns.fig   | (d) S-Par.fig              |
| (b) Tri.fig       | (e) E-S-Par.fig (Optional) |
| (c) Per-Nonin.fig |                            |