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Abstract Algebra
Day 30 Class Work

In the problems below, $F[x]$ is a polynomial ring where F is a field.

1. Let $f(x) = 2x + 7 \in \mathbb{R}[x]$.
   (a) Is $f(x)$ factorable or unfactorable in $\mathbb{R}[x]$? Explain your reasoning. **Ans:** Unfactorable.
   (b) Find $\alpha \in \mathbb{R}$ such that $f(\alpha) = 0$.
2. (a) Is $f(x) = 4x + 2$ factorable or unfactorable in $\mathbb{Z}_7[x]$?
   (b) Is $f(x) = ax + b$ with $a \neq 0$ factorable or unfactorable in $F[x]$? ← i.e., $\deg f(x) = 1$.
3. Let $f(x) = x^3 + x + 1 \in \mathbb{Z}_3[x]$.
   (a) Compute $f(\alpha)$ for each $\alpha \in \mathbb{Z}_3$.
   (b) Use your work in part (a) to determine if $f(x)$ is factorable or unfactorable in $\mathbb{Z}_3[x]$. Explain your reasoning. **Ans:** Factorable.
4. Let $f(x) = x^{5217} + 100 \in \mathbb{Z}_{101}[x]$.
   (a) Find $\alpha \in \mathbb{Z}_{101}$ such that $f(\alpha) = 0$.
   (b) Is $f(x)$ factorable or unfactorable in $\mathbb{Z}_{101}[x]$? Explain your reasoning.

Definition. An element $\alpha \in F$ is a *root* of a polynomial $f(x)$ if $f(\alpha) = 0$. ← For example, $\alpha = 1$ is a root of $f(x)$ in Prob. #4.

5. Let $f(x) \in F[x]$ with $\deg f(x) \geq 2$.
   (a) **Prove:** If $f(x)$ has a root $\alpha \in F$, then $f(x)$ is factorable in $F[x]$.
   (b) Why is the condition $\deg f(x) \geq 2$ needed here? **Hint:** See Problem #1.
6. Let $f(x) \in F[x]$ with $\deg f(x) \geq 2$.
   (a) **True or False:** If $f(x)$ has no root in F, then $f(x)$ is unfactorable in $F[x]$.
   (b) Let $f(x) = x^4 + 3x^2 + 2 \in \mathbb{R}[x]$. Explain why $f(\alpha) > 0$ for all $\alpha \in \mathbb{R}$. ← So, $f(\alpha)$ never equals 0.
   (c) Factor $f(x) = x^4 + 3x^2 + 2$ in $\mathbb{R}[x]$. **Ans:** $(x^2 + 1) \cdot (x^2 + 2)$.
   (d) If necessary, go back and revisit part (a) of this question.
7. Here's a salvage to Problem #6: **Assume $\deg f(x) = 2$ or 3.**
   (a) **Prove:** If $f(x)$ has no root in F, then $f(x)$ is unfactorable in $F[x]$.
   Hint: Try to prove its contrapositive.
   (b) Why was it necessary that $\deg f(x) = 2$ or 3? Where did you use that in your proof?
   (c) Is the polynomial in Problem #6(b) a counter-example? Why or why not?

8. Determine if each polynomial is factorable or unfactorable.
   (a) $x^{100} - 1$ in $\mathbb{R}[x]$
   (b) $x^3 + x + 1$ in $\mathbb{Z}_5[x]$
   (c) $x^2 + 1$ in $\mathbb{Z}_{13}[x]$
   (d) $x^{1071} + 10x^{282} + 4x^{123} + 2$ in $\mathbb{Z}_{17}[x]$
   (e) $x^2 + x + 4$ in $\mathbb{Z}_{11}[x]$

9. Consider $f(x) = x^4 - 1 \in \mathbb{Z}_5[x]$.
   (a) Elizabeth says, "I can tell right away that $f(x)$ is factorable." How might she know?
   (b) Compute the product
   $$(x-1)(x-2)(x-3)(x-4)$$
   in $\mathbb{Z}_5[x]$ and verify that it equals $f(x)$.

10. (a) How do you think $x^6 - 1$ would factor in $\mathbb{Z}_7[x]$? Verify your conjecture.
   (b) How do you think $x^{10} - 1$ would factor in $\mathbb{Z}_{11}[x]$? Verify your conjecture.
   (c) How do you think $x^{12} - 1$ would factor in $\mathbb{Z}_{13}[x]$? Yes, please verify.
   (d) What's going on here? Can you generalize and justify?

11. (a) In $\mathbb{Z}_2[x]$, find the number of polynomials of the form $x^2 + bx + c$, with b, $c \in \mathbb{Z}_2$. ← The coefficient of x^2 is 1.
   (b) Of the polynomials in part (a), determine how many of them are factorable.

12. Repeat Problem #11 in $\mathbb{Z}_3[x]$; in $\mathbb{Z}_5[x]$; in $\mathbb{Z}_7[x]$; in $\mathbb{Z}_{11}[x]$; in $\mathbb{Z}_p[x]$ where p is prime.