Abstract Algebra
Day 21 Class Work Solutions

1. With $H = \{1, 3, 9\}$, consider the set of cosets $U_{13}/H = \{1H,\ 2H,\ 4H,\ 7H\}$.

   Note: For reference, the cosets of H are...

   - $1H = 3H = 9H = \{1, 3, 9\}$ (original subgroup)
   - $2H = 5H = 6H = \{2, 5, 6\}$
   - $4H = 10H = 12H = \{4, 10, 12\}$
   - $7H = 8H = 11H = \{7, 8, 11\}$

   (a) Compute $4H \cdot 7H$ by multiplying each element of $4H$ by those of $7H$.

   $$4H \cdot 7H = \{4, 12, 10\} \cdot \{7, 8, 11\}$$
   $$= \{ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \}$$
   $$= \{ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \}$$

   Solution.

   $$\begin{aligned} 4H \cdot 7H &= \{4, 12, 10\} \cdot \{7, 8, 11\} \\ &= \{4\cdot 7,\ 4\cdot 8,\ 4\cdot 11,\ 12\cdot 7,\ 12\cdot 8,\ 12\cdot 11,\ 10\cdot 7,\ 10\cdot 8,\ 10\cdot 11\} \\ &= \{2, 6, 5, 6, 5, 2, 5, 2, 6\} \\ &= 2H \end{aligned}$$

   (b) Anita claims that even without computing the coset product, she knows $7H \cdot 4H$ would equal $4H \cdot 7H$. How does she know?

   Solution. The set $4H \cdot 7H$ contains elements of the form $a \cdot b$ where $a \in 4H$ and $b \in 7H$. Since multiplication in U_{13} is commutative, we have $a \cdot b = b \cdot a$. Thus, the set $7H \cdot 4H$ will contain the same elements as $4H \cdot 7H$.

   (c) Create a group table for U_{13}/H. Compute more coset products as needed.

$\cdot$	$1H$	$2H$	$4H$	$7H$
$1H$				
$2H$			$7H$	
$4H$				
$7H$				

   Solution.

$\cdot$	$1H$	$2H$	$4H$	$7H$
$1H$	$1H$	$2H$	$4H$	$7H$
$2H$	$2H$	$4H$	$7H$	$1H$
$4H$	$4H$	$7H$	$1H$	$2H$
$7H$	$7H$	$1H$	$2H$	$4H$

   (d) Verify that U_{13}/H is a group. Is it commutative or non-commutative?

   ← The operation of U_{13}/H is coset multiplication.

   Note: You may assume that coset multiplication is associative.

   Solution. U_{13}/H is a *commutative* group.

   - U_{13}/H is closed under coset multiplication. We can see this from the table, since every entry in the table (i.e., all possible "products") is an element of U_{13}/H.

- Coset multiplication is associative. (Assumed for now.)
- U_{13}/H contains the identity element $1H$. (Do you see why it's the identity?)
- Every element in U_{13}/H has an inverse. $2H$ and $7H$ are inverses of each other, and $1H$ and $4H$ are self inverses.

(e) Find the order of each $aH \in U_{13}/H$. Is the group cyclic?

Solution. Using the table, we can compute the order of each $aH \in U_{13}/H$, i.e., the number of times we multiply aH by itself to obtain the identity $1H$.

- $\text{ord}(1H) = 1$, because $(1H)^1 = 1H$.
- $\text{ord}(2H) = 4$, because $(2H)^1 = 2H$, $(2H)^2 = 4H$, $(2H)^3 = 7H$, and $(2H)^4 = 1H$.
- $\text{ord}(4H) = 2$, because $(4H)^1 = 4H$, and $(4H)^2 = 1H$.
- $\text{ord}(7H) = 4$, because $(7H)^1 = 7H$, $(7H)^2 = 4H$, $(7H)^3 = 2H$, and $(7H)^4 = 1H$.

Therefore, U_{13}/H is a *cyclic* group with generators $2H$ and $7H$.

2. Elizabeth claims she can compute $4H \cdot 7H$ *without* multiplying each element of $4H$ by those of $7H$. How? Try to *justify* her claim.

Solution. She found a shortcut, namely $4H \cdot 7H = (4 \cdot 7)H$. Note that $4 \cdot 7 = 2 \pmod{13}$, so that $(4 \cdot 7)H = 2H$, which is what we found in Problem #1(a).

3. Let $H = \{\varepsilon, v\}$ be a subgroup of D_4.

Note: See the next page for the group table for D_4.

(a) Compute the cosets $r_{90}H$ and $d'H$.

Solution. We have $r_{90}H = \{r_{90}, d\}$ and $d'H = \{d', r_{270}\}$.

(b) Compute the coset product $r_{90}H \cdot d'H$. You should actually compute the product, rather than using any shortcut that you may have found.

← Multiply each element of $r_{90}H$ by those of $d'H$.

Solution. We have...

$$\begin{aligned} r_{90}H \cdot d'H &= \{r_{90}, d\} \cdot \{d', r_{270}\} \\ &= \{r_{90}d', r_{90}r_{270}, d\,d', d\,r_{270}\} \\ &= \{v, \varepsilon, r_{180}, h\} \end{aligned}$$

This product is *not* a coset of H (it has too many elements).

(c) Does Elizabeth's shortcut for coset multiplication work here? What's going on?!

Ans: No. (Why not?)

Solution. Since $(r_{90} \cdot d')H = vH = H$, we have $r_{90}H \cdot d'H \neq (r_{90} \cdot d')H$. In other words, coset multiplication shortcut fails!

Here's the group table for D_4. Recall that for $\sigma, \tau \in D_4$, the "product" $\sigma \circ \tau$ is the entry in row σ and column τ. For example, the product $d \circ r_{90} = v$ is shown in bold.

$\circ$	ε	$\boldsymbol{r_{90}}$	r_{180}	r_{270}	h	v	d	d'
ε	ε	r_{90}	r_{180}	r_{270}	h	v	d	d'
r_{90}	r_{90}	r_{180}	r_{270}	ε	d'	d	h	v
r_{180}	r_{180}	r_{270}	ε	r_{90}	v	h	d'	d
r_{270}	r_{270}	ε	r_{90}	r_{180}	d	d'	v	h
h	h	d	v	d'	ε	r_{180}	r_{90}	r_{270}
v	v	d'	h	d	r_{180}	ε	r_{270}	r_{90}
$\boldsymbol{d}$	d	$\boldsymbol{v}$	d'	h	r_{270}	r_{90}	ε	r_{180}
d'	d'	h	d	v	r_{90}	r_{270}	r_{180}	ε

4. Consider the subgroup $Z = \{\varepsilon, r_{180}\}$ of D_4. Recall that Z is the *center* of D_4, i.e., the set of elements of D_4 that commute with all elements of D_4.

   (a) Quick, how many distinct left cosets of Z are there? Explain how you know.

   Solution. D_4 and Z contain 8 and 2 elements, respectively. Therefore, there are $8 \div 2 = 4$ distinct left cosets of Z.

   (b) Find the left coset aZ for each $a \in D_4$, and verify your answer from part (a).

   Solution. Indeed, there are 4 left cosets of Z, as shown below:
   - $\varepsilon Z = r_{180}Z = \{\varepsilon, r_{180}\}$
   - $r_{90}Z = r_{270}Z = \{r_{90}, r_{270}\}$
   - $hZ = vZ = \{h, v\}$
   - $dZ = d'Z = \{d, d'\}$

   (c) Verify that you found $D_4/Z = \{\varepsilon Z,\ r_{90}Z,\ hZ,\ dZ\}$ in part (b).

   Note: Using, for example, hZ instead of vZ is simply a matter of choice. ← They're the same coset.

   (d) Compute the coset product $r_{90}Z \cdot hZ$:

   $$r_{90}Z \cdot hZ = \{r_{90}, r_{270}\} \cdot \{h, v\}$$ ← Don't use the shortcut!

   $$= \{ \qquad \qquad \qquad \qquad \}$$

   $$= \{ \qquad \qquad \qquad \qquad \}$$

   Then verify that this product is indeed equal to $(r_{90} \cdot h)Z$.

   Solution. We have

   $$r_{90}Z \cdot hZ = \{r_{90}, r_{270}\} \cdot \{h, v\} = \{r_{90}h, r_{90}v, r_{270}h, r_{270}v\} = \{d', d, d, d'\} = \{d, d'\}.$$

   Moreover, $(r_{90} \cdot h)Z = d'Z = \{d'\varepsilon, d'r_{180}\} = \{d', d\}$. Thus, $r_{90}Z \cdot hZ = (r_{90} \cdot h)Z$.

   (e) Same as part (d), but now verify that $\varepsilon Z \cdot dZ = (\varepsilon \cdot d)Z$.

   Solution. We have

   $$\varepsilon Z \cdot dZ = \{\varepsilon, r_{180}\} \cdot \{d, d'\} = \{\varepsilon d, \varepsilon d', r_{180}d, r_{180}d'\} = \{d, d', d', d\} = \{d, d'\}.$$

   Moreover, $(\varepsilon \cdot d)Z = dZ = \{d\varepsilon, dr_{180}\} = \{d, d'\}$. Thus, $\varepsilon Z \cdot dZ = (\varepsilon \cdot d)Z$.

5. Let G be a group, H a subgroup, and $a, b \in G$. Define the coset product by

   $$aH \cdot bH = \{\alpha \cdot \beta \mid \alpha \in aH, \beta \in bH\}.$$

   (a) Under which condition(s) does the equality $aH \cdot bH = (ab)H$ hold? Look back at your work in Problems #1, #3, and #4 to come up with a conjecture. ← When does Elizabeth's shortcut work?

   (b) Try to prove your conjecture from part (a).

6. Consider the subgroup $H = \{1, 3, 9\}$ of U_{13}.

   (a) Verify that $7H = 8H$ and $8^{-1} \cdot 7 \in H$.

   Solution. We have $7H = 8H = \{7, 8, 11\}$. Also, $8^{-1} \cdot 7 = \ 5 \cdot 7 = 9 \in H$.

   (b) Verify that $7H \neq 4H$ and $4^{-1} \cdot 7 \notin H$.

   Solution. We have $7H = \{7, 8, 11\}$ and $4H = \{4, 12, 10\}$, and thus $7H \neq 4H$. Moreover, $4^{-1} \cdot 7 = 10 \cdot 7 = 5 \notin H$.

   (c) **Prove:** $aH = bH$ if and only if $b^{-1}a \in H$.

   Solution. This is Theorem 19.16 in the textbook.