Abstract Algebra
Day 21 Class Work

1. With $H = \{1, 3, 9\}$, consider the set of cosets $U_{13}/H = \{1H,\ 2H,\ 4H,\ 7H\}$.

   Note: For reference, the cosets of H are...

   - $1H = 3H = 9H = \{1, 3, 9\}$ (original subgroup)
   - $2H = 5H = 6H = \{2, 5, 6\}$
   - $4H = 10H = 12H = \{4, 10, 12\}$
   - $7H = 8H = 11H = \{7, 8, 11\}$

   (a) Compute $4H \cdot 7H$ by multiplying each element of $4H$ by those of $7H$.

   $4H \cdot 7H = \{4, 12, 10\} \cdot \{7, 8, 11\}$

   $= \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$

   $= \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$

   (b) Anita claims that even without computing the coset product, she knows $7H \cdot 4H$ would equal $4H \cdot 7H$. How does she know?

   (c) Create a group table for U_{13}/H. Compute more coset products as needed.

$\cdot$	$1H$	$2H$	$4H$	$7H$
$1H$				
$2H$			$7H$	
$4H$				
$7H$				

   (d) Verify that U_{13}/H is a group. Is it commutative or non-commutative?

   ← The operation of U_{13}/H is coset multiplication.

   Note: You may assume that coset multiplication is associative.

   (e) Find the order of each $aH \in U_{13}/H$. Is the group cyclic?

2. Elizabeth claims she can compute $4H \cdot 7H$ *without* multiplying each element of $4H$ by those of $7H$. How? Try to *justify* her claim.

3. Let $H = \{\varepsilon, v\}$ be a subgroup of D_4.

   Note: See the next page for the group table for D_4.

   (a) Compute the cosets $r_{90}H$ and $d'H$.

   (b) Compute the coset product $r_{90}H \cdot d'H$. You should actually compute the product, rather than using any shortcut that you may have found.

   ← Multiply each element of $r_{90}H$ by those of $d'H$.

   $r_{90}H \cdot d'H = \{r_{90}, d\} \cdot \{d', r_{270}\}$

   $= \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$

   $= \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$

   (c) Does Elizabeth's shortcut for coset multiplication work here? What's going on?!

   Ans: No. (Why not?)

Here's the group table for D_4. Recall that for $\sigma,\ \tau \in D_4$, the "product" $\sigma \circ \tau$ is the entry in row σ and column τ. For example, the product $d \circ r_{90} = v$ is shown in bold.

$\circ$	ε	$\boldsymbol{r_{90}}$	r_{180}	r_{270}	h	v	d	d'
ε	ε	r_{90}	r_{180}	r_{270}	h	v	d	d'
r_{90}	r_{90}	r_{180}	r_{270}	ε	d'	d	h	v
r_{180}	r_{180}	r_{270}	ε	r_{90}	v	h	d'	d
r_{270}	r_{270}	ε	r_{90}	r_{180}	d	d'	v	h
h	h	d	v	d'	ε	r_{180}	r_{90}	r_{270}
v	v	d'	h	d	r_{180}	ε	r_{270}	r_{90}
$\boldsymbol{d}$	d	$\boldsymbol{v}$	d'	h	r_{270}	r_{90}	ε	r_{180}
d'	d'	h	d	v	r_{90}	r_{270}	r_{180}	ε

4. Consider the subgroup $Z = \{\varepsilon,\ r_{180}\}$ of D_4. Recall that Z is the *center* of D_4, i.e., the set of elements of D_4 that commute with all elements of D_4.

   (a) Quick, how many distinct left cosets of Z are there? Explain how you know.

   (b) Find the left coset aZ for each $a \in D_4$, and verify your answer from part (a).

   (c) Verify that you found $D_4/Z = \{\varepsilon Z,\ r_{90}Z,\ hZ,\ dZ\}$ in part (b).

   Note: Using, for example, hZ instead of vZ is simply a matter of choice. ← They're the same coset.

   (d) Compute the coset product $r_{90}Z \cdot hZ$:

   $r_{90}Z \cdot hZ = \{r_{90},\ r_{270}\} \cdot \{h,\ v\}$ ← Don't use the shortcut!

   $= \{ \qquad \}$

   $= \{ \qquad \}$

   Then verify that this product is indeed equal to $(r_{90} \cdot h)Z$.

   (e) Same as part (d), but now verify that $\varepsilon Z \cdot dZ = (\varepsilon \cdot d)Z$.

5. Let G be a group, H a subgroup, and $a,\ b \in G$. Define the coset product by

$$aH \cdot bH = \{\alpha \cdot \beta \mid \alpha \in aH,\ \beta \in bH\}.$$

   (a) Under which condition(s) does the equality $aH \cdot bH = (ab)H$ hold? Look back at your work in Problems #1, #3, and #4 to come up with a conjecture. ← When does Elizabeth's shortcut work?

   (b) Try to prove your conjecture from part (a).

6. Consider the subgroup $H = \{1,\ 3,\ 9\}$ of U_{13}.

   (a) Verify that $7H = 8H$ and $8^{-1} \cdot 7 \in H$.

   (b) Verify that $7H \neq 4H$ and $4^{-1} \cdot 7 \notin H$.

   (c) **Prove:** $aH = bH$ if and only if $b^{-1}a \in H$.