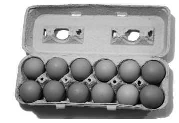
Abstract Algebra
Day 20 Class Work

1. (a) There are lots of eggs and dozen egg cartons. If all the eggs are in cartons and all the cartons are full, can there be 1000 eggs? Why or why not?

   ← This kind of carton:

   (b) Let G be a group. If a subgroup H has 12 elements, can the group G contain 1000 elements? Why or why not?

2. Let H be a subgroup of a *finite* group G. In answering these, try to *justify* your claims.

   (a) Suppose $\#G = 28$ and $\#H = 4$, where $\#G$ and $\#H$ denote the sizes of G and H, respectively. Find $[G : H]$, i.e., the number of distinct left cosets of H.

   Ans: $[G : H] = 7$.

   (b) Can a group with 28 elements have a subgroup of size 5? Why or why not? Give an explanation using cosets.

   (c) Find a general formula for $[G : H]$. Explain your reasoning.

   (d) Explain why $\#H$ is a divisor of $\#G$.

3. Let G be a finite group, and consider an element $g \in G$ with $\text{ord}(g) = 6$.

   (a) Let $\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\}$ be the cyclic subgroup generated by g. Write down the *distinct* elements of $\langle g \rangle$. How many elements does $\langle g \rangle$ contain?

   (b) Can the group G contain 34 elements? Why or why not?

   Ans: No. (Why not?)

   Hint: Apply Problem #2(d) with $H = \langle g \rangle$.

4. **Prove:** Let G be a finite group and $g \in G$. Then $\text{ord}(g)$ is a divisor of $\#G$.

   Hint: See Problem #3.

5. Suppose a group G contains 5 elements, and let $g \in G$ be a non-identity element.

   (a) Find $\text{ord}(g)$.

   Ans: $\text{ord}(g) = 5$.

   (b) How many elements does the cyclic subgroup $\langle g \rangle$ contain?

   (c) Explain why G is cyclic with generator g.

6. Repeat Problem #5 with a group G that contains 7 elements; 19 elements; 101 elements; p elements where p is prime.

7. Consider the group D_4 and its subgroup $H = \{\varepsilon, r_{180}, d, d'\}$.

   Note: You should be able to complete this problem *without* the table for D_4.

   (a) Find $[D_4 : H]$.

   Ans: $[D_4 : H] = 2$.

   (b) Suppose $a \in H$. Determine the elements in the coset aH.

   (c) Same as part (b), but with $a \notin H$.

8. In this problem, you'll prove that the distinct cosets of H form a *partition* of G, i.e.,

   - they cover all of G, and
   - they do not overlap with each other.

   (a) Give an example that illustrates this notion of a partition.

(b) **Prove:** Every element of G is contained in some coset of H. ← i.e., they cover all of G.

(c) **Prove:** If $aH \neq bH$, then aH and bH do not have any element in common. ← i.e., they don't overlap.
Hint: Think contrapositive.

9. Let G be a group and H and K its subgroups. Define $M = \{g \in G \mid g \in H \text{ and } g \in K\}$. ← i.e., M is the *intersection* of H and K.

   (a) **Prove:** M is a subgroup of G.

   (b) If $\#H = 21$ and $\#K = 32$, find $\#M$. Explain your reasoning.

10. Consider the prime number $p = 3$.

    (a) Choose an integer a, compute $a^p - a$, and verify that p is a divisor of $a^p - a$.

    (b) Repeat part (a) with another integer a of your choice.

    (c) Repeat part (a) again, this time with a negative integer a.

11. (a) Repeat Problem #10 with prime $p = 5$; with prime $p = 7$; with prime $p = 11$.

    (b) Repeat Problem #10 with one more prime number of your choice.

    (c) What conjecture do you have?

12. **Prove:** Let p be a prime number. Prove that p is a divisor of $a^p - a$ for all $a \in \mathbb{Z}$. ← This is called *Fermat's little theorem*.