Abstract Algebra
Day 17 Class Work Solutions

Below, we will look at the following functions:

- $\varphi : \mathbb{Z} \to \mathbb{Z}_5$ where $\varphi(a) = a \pmod 5$ for all $a \in \mathbb{Z}$.
- $\gamma : \mathbb{Z}_{12} \to \mathbb{Z}_{18}$ where $\gamma(a) = 6a$ for all $a \in \mathbb{Z}_{12}$.
- $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$.
- $\delta : G(\mathbb{Z}_{10}) \to U_{10}$ where $\delta(\alpha) = \det \alpha$ for all $\alpha \in G(\mathbb{Z}_{10})$.

Recall: $G(\mathbb{Z}_{10})$ contains matrices with mult. inverses.

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1. Consider the function γ.

   (a) Compute $\gamma(7 + 10)$ and $\gamma(7) + \gamma(10)$ and verify that they're equal.

   Ans: 12 for both.

   Solution. We have $\gamma(7 + 10) = \gamma(5) = 30 = 12$. Note here that the sum $7 + 10$ is computed in $\mathbb{Z}_{12}$, and the reduction $30 = 12$ is done in $\mathbb{Z}_{18}$. We also have $\gamma(7) + \gamma(10) = 42 + 60 = 102 = 12$, where the reduction $102 = 12$ is done in $\mathbb{Z}_{18}$. Thus, both sides are equal to 12 (in $\mathbb{Z}_{18}$).

   (b) Show that $\gamma(a + b) = \gamma(a) + \gamma(b)$ for all $a, b \in \mathbb{Z}_{12}$. (Thus, γ is a homomorphism.)

   Solution. We have $\gamma(a + b) = 6(a + b) = 6a + 6b = \gamma(a) + \gamma(b)$. The key step $6(a + b) = 6a + 6b$ is due to the distributive law.

2. Now consider the function λ.

   (a) Compute $\lambda(5 \cdot 2)$ and $\lambda(5) \cdot \lambda(2)$ and verify that they're equal.

   Ans: 12 for both again!

   Solution. We have $\lambda(5 \cdot 2) = \lambda(10) = 10^3 = 1000 = 12$, and $\lambda(5) \cdot \lambda(2) = 5^3 \cdot 2^3 = 125 \cdot 8 = 1000 = 12$. Thus, the both sides are equal to 12.

   (b) Show that $\lambda(a \cdot b) = \lambda(a) \cdot \lambda(b)$ for all $a, b \in U_{13}$. (Thus, λ is a homomorphism).

   Solution. We have $\lambda(a \cdot b) = (a \cdot b)^3 = a^3 \cdot b^3 = \lambda(a) \cdot \lambda(b)$. They key step $(a \cdot b)^3 = a^3 \cdot b^3$ is due to an exponent law, which holds here because multiplication in U_{13} is commutative.

3. Which familiar determinant property tells us that δ is also a homomorphism?

   Hint: $\det(\alpha \cdot \beta) = ?$

   Note: Recall that $G(\mathbb{Z}_{10})$ is a multiplicative group. What about U_{10}?

   Solution. For matrices $\alpha, \beta \in G(\mathbb{Z}_{10})$, we have

   $$\delta(\alpha \cdot \beta) = \det(\alpha \cdot \beta) = \det \alpha \cdot \det \beta = \delta(\alpha) \cdot \delta(\beta).$$

   The key determinant property is $\det(\alpha \cdot \beta) = \det \alpha \cdot \det \beta$, i.e., the determinant of the (matrix) product is equal to the product of the determinants.

4. (a) Anita says, "There's no way φ, γ, and δ are isomorphisms, because the elements don't match up." What might she mean?

   Solution. An isomorphism is a bijection, i.e., one-to-one and onto. Thus, the domain and codomain of a bijection must have the same number of elements. This is *not* the case in φ, where the domain $\mathbb{Z}$ has infinitely many elements and the codomain $\mathbb{Z}_5$ has five elements. For the same reason, neither γ nor δ is an isomorphism.

   (b) How about the function λ? Is it an isomorphism? Why or why not?

   Ans: It's not. (Why not?)

   Solution. While λ has the same domain and codomain, it is *not* an isomorphism. For instance, we have $\lambda(1) = \lambda(3) = \lambda(9) = 1$, so λ is *not* one-to-one (i.e., different inputs 1, 3, and 9 map to the same output 1). Note that λ isn't onto either, because not every element in the codomain U_{13} get "hit" by the function.

   ← 1, 5, 8, 12 get "hit."

5. Note that $\varphi(0) = 0 \pmod 5$, i.e., φ maps the identity of $\mathbb{Z}$ to the identity of $\mathbb{Z}_5$.

   (a) Verify that γ maps the identity of $\mathbb{Z}_{12}$ to the identity of $\mathbb{Z}_{18}$.

   Solution. $\gamma(0) = 6 \cdot 0 = 0$.

   (b) Verify that λ maps the identity of U_{13} to the identity of U_{13}.

   Solution. $\lambda(1) = 1^3 = 1$.

   (c) Verify that δ does the same.

   Solution. $\delta(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]) = \det(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]) = 1$.

   (d) Any conjectures?

   Solution. Let $\theta : G \to H$ be a group homomorphism. Then θ maps the identity of G to the identity of H, i.e., $\theta(\varepsilon_G) = \varepsilon_H$.

6. Consider the function φ.

   (a) How are 16 and -16 related in the domain $\mathbb{Z}$?

   Solution. We have $16 + (-16) = 0$ and $-16 + 16 = 0$. Therefore, 16 and -16 are (additive) inverses of each other in $\mathbb{Z}$.

   (b) Compute $\varphi(16)$ and $\varphi(-16)$. How are they related in the codomain $\mathbb{Z}_5$?

   Solution. We have $\varphi(16) = 1$ and $\varphi(-16) = 4$. Note that $1 + 4 = 0$ and $4 + 1 = 0$ in $\mathbb{Z}_5$, so that $\varphi(16)$ and $\varphi(-16)$ are (additive) inverses of each other in $\mathbb{Z}_5$.

7. Consider the function δ and the following elements of $G(\mathbb{Z}_{10})$:

$$\alpha = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} 8 & 3 \\ 5 & 4 \end{bmatrix}.$$

   (a) How are α and β related in the domain $G(\mathbb{Z}_{10})$? **Hint:** Compute $\alpha \cdot \beta$.

   Solution. We have $\alpha \cdot \beta = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\beta \cdot \alpha = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, so that α and β are (multiplicative) inverse of each other in $G(\mathbb{Z}_{10})$.

   (b) Compute $\delta(\alpha)$ and $\delta(\beta)$. How are they related in the codomain U_{10}? ← Think multiplicatively.

   Solution. We have $\delta(\alpha) = 3$ and $\delta(\beta) = 7$. Note that $3 \cdot 7 = 1$ and $7 \cdot 3 = 1$ in U_{10}, so that $\delta(\alpha)$ and $\delta(\beta)$ are (multiplicative) inverses of each other in U_{10}.

   (c) Any conjectures?

   Solution. Let $\theta : G \to H$ be a group homomorphism. If $x, y \in G$ are inverses of each other, then $\theta(x), \theta(y) \in H$ are inverses of each other.

8. Recall the function $\gamma : \mathbb{Z}_{12} \to \mathbb{Z}_{18}$ where $\gamma(a) = 6a$ for all $a \in \mathbb{Z}_{12}$. For order computations below, note that $\mathbb{Z}_{12}$ and $\mathbb{Z}_{18}$ are *additive* groups.

   (a) Find $\text{ord}(10)$ in $\mathbb{Z}_{12}$ and $\text{ord}(\gamma(10))$ in $\mathbb{Z}_{18}$. **Ans for (a):** 6 and 3.

   Solution. 6 and 3.

   (b) Find $\text{ord}(7)$ in $\mathbb{Z}_{12}$ and $\text{ord}(\gamma(7))$ in $\mathbb{Z}_{18}$.

   Solution. 12 and 3.

   (c) Find $\text{ord}(8)$ in $\mathbb{Z}_{12}$ and $\text{ord}(\gamma(8))$ in $\mathbb{Z}_{18}$.

   Solution. 3 and 3.

   (d) Find $\text{ord}(6)$ in $\mathbb{Z}_{12}$ and $\text{ord}(\gamma(6))$ in $\mathbb{Z}_{18}$.

Solution. 2 and 1.

(e) Any conjectures?

Solution. Let $\theta : G \to H$ be a group homomorphism. Then $\text{ord}(\theta(g))$ is a divisor of $\text{ord}(g)$ for all $g \in G$.

9. Consider $\delta : G(\mathbb{Z}_{10}) \to U_{10}$ where $\delta(\alpha) = \det \alpha$ for all $\alpha \in G(\mathbb{Z}_{10})$. Let $\alpha = \left[\begin{smallmatrix} 2 & 1 \\ 5 & 4 \end{smallmatrix}\right] \in G(\mathbb{Z}_{10})$.

   (a) Compute α^{-2} in two ways: via the interpretations $\alpha^{-2} = (\alpha^{-1})^2$ and $\alpha^{-2} = (\alpha^2)^{-1}$.

   (b) Use the result in part (a) to compute $\delta(\alpha^{-2})$.

   (c) Compute $\delta(\alpha)$ and use that to compute $\delta(\alpha)^{-2}$.

   (d) Compare $\delta(\alpha^{-2})$ with $\delta(\alpha)^{-2}$. Is the outcome surprising?

   Solution. You should find that $\delta(\alpha^{-2}) = \delta(\alpha)^{-2}$. (I'll leave the calculations up to you.) A generalization of this can be found in Problem #10(c) below.

10. Let $\theta : G \to H$ a group homomorphism. Prove each of the following. ← i.e., G and H are groups.

    (a) $\theta(\varepsilon_G) = \varepsilon_H$.

    Note: Here, ε_G and ε_H are identity elements of G and H, respectively.

    (b) $\theta(g^{-1}) = \theta(g)^{-1}$ for all $g \in G$.

    (c) $\theta(g^k) = \theta(g)^k$ for all $g \in G$ and $k \in \mathbb{Z}$.

    (d) $\text{ord}(\theta(g))$ is a divisor of $\text{ord}(g)$.

    Solution. See Chapter 17 for the proofs of these claims.