Abstract Algebra
Day 17 Class Work

Below, we will look at the following functions:

- $\varphi : \mathbb{Z} \to \mathbb{Z}_5$ where $\varphi(a) = a \pmod 5$ for all $a \in \mathbb{Z}$.
- $\gamma : \mathbb{Z}_{12} \to \mathbb{Z}_{18}$ where $\gamma(a) = 6a$ for all $a \in \mathbb{Z}_{12}$.
- $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$.
- $\delta : G(\mathbb{Z}_{10}) \to U_{10}$ where $\delta(\alpha) = \det \alpha$ for all $\alpha \in G(\mathbb{Z}_{10})$.

Recall: $G(\mathbb{Z}_{10})$ contains matrices with mult. inverses.

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1. Consider the function γ.
   (a) Compute $\gamma(7 + 10)$ and $\gamma(7) + \gamma(10)$ and verify that they're equal. **Ans:** 12 for both.
   (b) Show that $\gamma(a + b) = \gamma(a) + \gamma(b)$ for all $a, b \in \mathbb{Z}_{12}$. (Thus, γ is a homomorphism.)

2. Now consider the function λ.
   (a) Compute $\lambda(5 \cdot 2)$ and $\lambda(5) \cdot \lambda(2)$ and verify that they're equal. **Ans:** 12 for both again!
   (b) Show that $\lambda(a \cdot b) = \lambda(a) \cdot \lambda(b)$ for all $a, b \in U_{13}$. (Thus, λ is a homomorphism).

3. Which familiar determinant property tells us that δ is also a homomorphism? **Hint:** $\det(\alpha \cdot \beta) = ?$

   Note: Recall that $G(\mathbb{Z}_{10})$ is a multiplicative group. What about U_{10}?

4. (a) Anita says, "There's no way φ, γ, and δ are isomorphisms, because the elements don't match up." What might she mean?
   (b) How about the function λ? Is it an isomorphism? Why or why not? **Ans:** It's not. (Why not?)

5. Note that $\varphi(0) = 0 \pmod 5$, i.e., φ maps the identity of $\mathbb{Z}$ to the identity of $\mathbb{Z}_5$.
   (a) Verify that γ maps the identity of $\mathbb{Z}_{12}$ to the identity of $\mathbb{Z}_{18}$.
   (b) Verify that λ maps the identity of U_{13} to the identity of U_{13}.
   (c) Verify that δ does the same.
   (d) Any conjectures?

6. Consider the function φ.
   (a) How are 16 and -16 related in the domain $\mathbb{Z}$?
   (b) Compute $\varphi(16)$ and $\varphi(-16)$. How are they related in the codomain $\mathbb{Z}_5$?

7. Consider the function δ and the following elements of $G(\mathbb{Z}_{10})$:

$$\alpha = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} 8 & 3 \\ 5 & 4 \end{bmatrix}.$$

   (a) How are α and β related in the domain $G(\mathbb{Z}_{10})$? **Hint:** Compute $\alpha \cdot \beta$.
   (b) Compute $\delta(\alpha)$ and $\delta(\beta)$. How are they related in the codomain U_{10}? ← Think multiplicatively.
   (c) Any conjectures?

8. Recall the function $\gamma : \mathbb{Z}_{12} \to \mathbb{Z}_{18}$ where $\gamma(a) = 6a$ for all $a \in \mathbb{Z}_{12}$. For order computations below, note that $\mathbb{Z}_{12}$ and $\mathbb{Z}_{18}$ are *additive* groups.
   (a) Find $\text{ord}(10)$ in $\mathbb{Z}_{12}$ and $\text{ord}(\gamma(10))$ in $\mathbb{Z}_{18}$. **Ans for (a):** 6 and 3.
   (b) Find $\text{ord}(7)$ in $\mathbb{Z}_{12}$ and $\text{ord}(\gamma(7))$ in $\mathbb{Z}_{18}$.
   (c) Find $\text{ord}(8)$ in $\mathbb{Z}_{12}$ and $\text{ord}(\gamma(8))$ in $\mathbb{Z}_{18}$.
   (d) Find $\text{ord}(6)$ in $\mathbb{Z}_{12}$ and $\text{ord}(\gamma(6))$ in $\mathbb{Z}_{18}$.
   (e) Any conjectures?

9. Consider $\delta : G(\mathbb{Z}_{10}) \to U_{10}$ where $\delta(\alpha) = \det \alpha$ for all $\alpha \in G(\mathbb{Z}_{10})$. Let $\alpha = \left[\begin{smallmatrix} 2 & 1 \\ 5 & 4 \end{smallmatrix}\right] \in G(\mathbb{Z}_{10})$.
   (a) Compute α^{-2} in two ways: via the interpretations $\alpha^{-2} = (\alpha^{-1})^2$ and $\alpha^{-2} = (\alpha^2)^{-1}$.
   (b) Use the result in part (a) to compute $\delta(\alpha^{-2})$.
   (c) Compute $\delta(\alpha)$ and use that to compute $\delta(\alpha)^{-2}$.
   (d) Compare $\delta(\alpha^{-2})$ with $\delta(\alpha)^{-2}$. Is the outcome surprising?

10. Let $\theta : G \to H$ a group homomorphism. Prove each of the following. ← i.e., G and H are groups.
   (a) $\theta(\varepsilon_G) = \varepsilon_H$.
   Note: Here, ε_G and ε_H are identity elements of G and H, respectively.
   (b) $\theta(g^{-1}) = \theta(g)^{-1}$ for all $g \in G$.
   (c) $\theta(g^k) = \theta(g)^k$ for all $g \in G$ and $k \in \mathbb{Z}$.
   (d) $\text{ord}(\theta(g))$ is a divisor of $\text{ord}(g)$.