Abstract Algebra
Day 16 Class Work

1. Let g be a group element with $\text{ord}(g) = 6$. Consider the cyclic group $\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\}$. We just saw that $\langle g \rangle = \{g^0, g^1, g^2, g^3, g^4, g^5\}$, where $g^0 = \varepsilon$.

   ← i.e., $\langle g \rangle$ is the set of all integer powers of g.

   (a) Write down the distinct elements of the additive group $\mathbb{Z}_6$.

   (b) How do the elements in $\mathbb{Z}_6$ and $\langle g \rangle$ "match up"?

   (c) Compute the following and describe how they're related:

   - $3 + 5$ in $\mathbb{Z}_6$ and $g^3 \cdot g^5$ in $\langle g \rangle$.
   - -2 in $\mathbb{Z}_6$ and $(g^2)^{-1}$ in $\langle g \rangle$.

   (d) How do $\mathbb{Z}_6$ and $\langle g \rangle$ behave similarly as groups?

   ← Compare addition in $\mathbb{Z}_6$ & multiplication in $\langle g \rangle$.

2. Let g be a group element with $\text{ord}(g) = 6$. Consider the function $\theta : \mathbb{Z}_6 \to \langle g \rangle$ where $\theta(a) = g^a$ for all $a \in \mathbb{Z}_6$. For example, $\theta(3) = g^3$.

   (a) Make sense of the equation: $\theta(3 + 5) = \theta(3) \cdot \theta(5)$.

   (b) How does the equation in part (a) capture the notion that the operations of $\mathbb{Z}_6$ and $\langle g \rangle$ "match up"?

3. Let g be a group element with *infinite* order. Consider the function $\theta : \mathbb{Z} \to \langle g \rangle$ where $\theta(a) = g^a$ for all $a \in \mathbb{Z}$.

   ← Or choose $g = 3$ in the multiplicative group $\mathbb{R}^*$.

   (a) Find $a \in \mathbb{Z}$ such that $\theta(a) = g^{-34}$.

   (b) Show that θ is onto.

   (c) Explain why $g^{12} \neq g^7$.

   Hint: Assume $g^{12} = g^7$ and obtain a contradiction.

   (d) Show that θ is one-to-one.

   Note: Since $\text{ord}(g) = \infty$, there is no positive integer n such that $g^n = \varepsilon$.

   ← In other words, $g^n = \varepsilon$ would imply that $n = 0$.

   (e) Show that $\theta(a + b) = \theta(a) \cdot \theta(b)$ for all $a, b \in \mathbb{Z}$.

   Note: Thus, we say that θ is *operation preserving*.

4. Let G and H be groups. Consider $\theta : G \to H$ where $\theta(a \cdot b) = \theta(a) \cdot \theta(b)$ for all $a, b \in G$.

   (a) In the equation $\theta(a \cdot b) = \theta(a) \cdot \theta(b)$, there are two products: $a \cdot b$ and $\theta(a) \cdot \theta(b)$. In which group, G or H, does each of these multiplications take place?

   Ans: $a \cdot b$ occurs in G, and $\theta(a) \cdot \theta(b)$ happens in H.

   (b) Suppose θ is one-to-one. Prove that if H is commutative, then G is commutative.

   (c) Suppose θ is onto. Prove that if G is commutative, then H is commutative.

5. Recall that $\mathbb{R}$ is the additive group of all real numbers. Define $\mathbb{R}^{>0} = \{r \in \mathbb{R} \mid r > 0\}$, i.e., the set of all *positive* real numbers.

   (a) Explain why $\mathbb{R}^{>0}$ is a group under multiplication.

   (b) Define a function $\alpha : \mathbb{R} \to \mathbb{R}^{>0}$ where $\alpha(x) = 3^x$ for all $x \in \mathbb{R}$. Show that α is one-to-one and onto.

   (c) Show that $\alpha(x + y) = \alpha(x) \cdot \alpha(y)$ for all $x, y \in \mathbb{R}$.

6. Consider again the additive group $\mathbb{R}$, and also the multiplicative group $\mathbb{R}^*$ of nonzero real numbers. Explain why there cannot be a function $\theta : \mathbb{R} \to \mathbb{R}^*$ that is one-to-one and onto, and satisfies $\theta(x + y) = \theta(x) \cdot \theta(y)$ for all $x, y \in \mathbb{R}$.

   Hint: If such a function exists, then $\theta(r) = -1$ for some $r \in \mathbb{R}$. Explain why that would lead to a contradiction.

7. **(Some Food for Thought)** Let $S = \{a, b, c, d, e\}$ and $T = \{x, y, z\}$.

   (a) How many different *onto* functions are there from S to T?

   (b) What if S and T have m and n elements, respectively?