Abstract Algebra
Day 14 Class Work

1. Determine if each of these groups is cyclic. ← Two of them are cyclic.

   (a) Additive group $\mathbb{Z}_{12}$.

   (b) Multiplicative group U_{13}.

   (c) Multiplicative group $\mathbb{R}^*$.

2. (a) Find all generators of $\mathbb{Z}_{12}$.

   (b) **(Review)** Recall that 2 is a generator of U_{13}, where ← and $2^{12} = 2^0 = 1$.

$$\begin{aligned} U_{13} &= \langle 2 \rangle \\ &= \{2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}, 2^{11}\} \\ &= \{1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7\} \end{aligned}$$

   - Find $9 + 7$ in $\mathbb{Z}_{12}$; and find k (with $0 \leq k \leq 11$) such that $2^9 \cdot 2^7 = 2^k$ in U_{13}.
   - Solve $4 + \boxed{?} = 0$ in $\mathbb{Z}_{12}$; and find k such that $2^4 \cdot 2^k = 1$ in U_{13}.

   (c) Use your results from parts (a) and (b) to find all other generators of U_{13}. **Ans:** 6, 11, 7.

3. (a) Find all subgroups of $\mathbb{Z}_{12}$. Are they cyclic, too? ← There are six of them.

   (b) Same as part (a), but with U_{13}.

   (c) What conjecture do you have?

4. We've seen that the *additive* group $\mathbb{Z}_{12}$ is cyclic, since $\mathbb{Z}_{12} = \langle 1 \rangle$. Our friends are discussing the meaning of $\langle 1 \rangle$ when the operation is addition:

   Elizabeth: I think $\langle 1 \rangle$ means $\{1^k \mid k \in \mathbb{Z}\}$.

   Anita: I disagree. I think $\langle 1 \rangle$ means $\{k \cdot 1 \mid k \in \mathbb{Z}\}$.

   With whom do you agree? Explain.

5. Explain why the additive group $\mathbb{Z}$ is cyclic. Find all of its generators. ← There are two generators.

6. (a) Verify that 4 and 10 are multiplicative inverses of each other in U_{13}. Then compute the cyclic subgroups $\langle 4 \rangle$ and $\langle 10 \rangle$. How do they compare?

   (b) Recall again the subgroup $\langle 3 \rangle \subseteq \mathbb{R}^*$. Now consider the cyclic subgroup $\left\langle \frac{1}{3} \right\rangle \subseteq \mathbb{R}^*$. In particular, how do $\langle 3 \rangle$ and $\left\langle \frac{1}{3} \right\rangle$ compare? Explain your reasoning. ← So, $\langle \frac{1}{3} \rangle$ is the set of all integer powers of $\frac{1}{3}$.

   (c) Let g be a group element. Prove that $\langle g \rangle = \langle g^{-1} \rangle$.

   Note: This is a set equality. So you must show $\langle g \rangle \subseteq \langle g^{-1} \rangle$ and $\langle g^{-1} \rangle \subseteq \langle g \rangle$.

7. Recall the subgroup $\langle 3 \rangle$ of the multiplicative group $\mathbb{R}^*$.

   (a) Find $k \in \mathbb{Z}$ such that $3^{17} \cdot 3^{25} = 3^k$ in $\langle 3 \rangle$.

   (b) Find $k \in \mathbb{Z}$ such that 3^k is the multiplicative identity of $\langle 3 \rangle$.

   (c) Find $k \in \mathbb{Z}$ such that 3^k is the multiplicative inverse of 3^{17} in $\langle 3 \rangle$.

   (d) Elizabeth says that $\langle 3 \rangle$ behaves just like the additive group $\mathbb{Z}$. What might she mean? Be as precise as possible.

8. (a) Find all subgroups of $\mathbb{Z}_{18}$.

   (b) Find all subgroups of U_{19}. (**Hint:** Find its generator first.)

9. Consider the additive group $\mathbb{Z}_{40}$.

   (a) Find a subgroup H of $\mathbb{Z}_{40}$ containing 10 elements.

   (b) Verify that H is cyclic by finding a generator.

   (c) Find *all* generators of H.

10. **Prove:** Let G be a cyclic group, and H a subgroup of G. Then H is also cyclic.