Opening experiment

Table for H:

$*$	1	3	9
1	1	3	9
3	3	9	1
9	9	1	3

Consider the (multiplicative) group

$$U_{13} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

and a subgroup $H = \{1, 3, 9\}$. (See table.)

Let $6 \in U_{13}$. We then have a coset

$$6H = \{6 \cdot 1, 6 \cdot 3, 6 \cdot 9\} = \{6, 5, 2\}.$$

Note: The coset $6H$ is *not* a subgroup of U_{13}. (Why not?)

Additive group example

Consider the (additive) group

$$\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

and a subgroup $H = \{0, 4, 8\}$.

Let $6 \in \mathbb{Z}_{12}$. We then have a coset

$$6 + H = \{6+0,\ 6+4,\ 6+8\} = \{6,\ 10,\ 2\}.$$

Note: Cosets of additive groups will be important when we study *rings*.

Examples of cosets:

- Group U_{13}, subgroup $H = \{1, 3, 9\}$, and coset $6H = \{6, 5, 2\}$.
- Group $\mathbb{Z}_{12}$, subgroup $H = \{0, 4, 8\}$, and coset $6 + H = \{6, 10, 2\}$.

Definition. Let G be a group, H a subgroup, and $a \in G$. Then

$$\text{(Multiplicative) } aH = \{ah \mid h \in H\}$$

$$\text{(Additive) } a + H = \{a + h \mid h \in H\}$$

is the left coset of H generated by a.

Note: The element a is called the *coset representative* of aH (or $a + H$).

Properties of a (left) coset

Cosets of $H = \{1, 3, 9\}$ in U_{13} are...

- $1H = 3H = 9H = \{1, 3, 9\}$
- $2H = 5H = 6H = \{2, 5, 6\}$
- $4H = 10H = 12H = \{4, 10, 12\}$
- $7H = 8H = 11H = \{7, 8, 11\}$

Cosets of $H = \{0, 4, 8\}$ in $\mathbb{Z}_{12}$ are...

- $0 + H = 4 + H = 8 + H = \{0, 4, 8\}$
- $1 + H = 5 + H = 9 + H = \{1, 5, 9\}$
- $2 + H = 6 + H = 10 + H = \{2, 6, 10\}$
- $3 + H = 7 + H = 11 + H = \{3, 7, 11\}$

Let G be a group and H a subgroup. Then...

(a) Let $a \in G$. Then...

(Multiplicative) a is in the coset aH, i.e., $a \in aH$.

(Additive) a is in the coset $a + H$, i.e., $a \in a + H$.

Reason:

$H = \{\varepsilon, \ldots\} \implies aH = \{a\varepsilon, \ldots\}$

$\implies a \in aH$.

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- $2 + H = 6 + H = 10 + H = \{2, 6, 10\}$
- $3 + H = 7 + H = 11 + H = \{3, 7, 11\}$

Let G be a group and H a subgroup. Then...

(b) Let $a \in G$. Then...

(Multiplicative) $aH = H$ if and only if $a \in H$.

(Additive) $a + H = H$ if and only if $a \in H$.

For a proof, see Chapter 19 reading.

Properties of a (left) coset

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Cosets of $H = \{0, 4, 8\}$ in $\mathbb{Z}_{12}$ are...

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Let G be a group and H a subgroup. Then...

(c) All the left cosets of H have the same size. (Proof in Chapter 19.)

(d) The distinct left cosets of H form a *partition* of G. (More next time!)

When are cosets equal?

Example 1. With $\mathbb{Z}_{12}$ and subgroup $H = \{0, 4, 8\}$: $3 + H = 11 + H$.

Example 2. With U_{13} and subgroup $H = \{1, 3, 9\}$: $2H = 6H$.

Problem #5: Let G be a group, H a subgroup, and $a, b \in G$.

Describe how a and b must be related so that...

- $a + H = b + H$ (for additive groups) $\leftarrow$ relationship is *additive*.
- $aH = bH$ (for multiplicative groups) $\leftarrow$ relationship is *multiplicative*.

$3 + H = 11 + H$ $\quad$ $3 - 11 \in H$ $\quad$ $11 - 3 \in H$

Additive group: $a + H = b + H \iff a - b \in H$ and $b - a \in H$.

Multiplicative group: First, a mnemonic, which is *not* a proof...

$$aH = bH \overset{*b^{-1}}{\iff} b^{-1} \cdot a H = H \iff b^{-1} \cdot a \in H. \quad \text{(Likewise for } a^{-1} \cdot b \in H.\text{)}$$

Thus, $aH = bH \iff b^{-1} \cdot a \in H$ and $a^{-1} \cdot b \in H$. (Not $a \cdot b^{-1}$, $b \cdot a^{-1} \in H$.)

Examples: With U_{13} and $H = \{1, 3, 9\}$,

- $\overset{a}{2}H = \overset{b}{6}H \iff 6^{-1} \cdot 2 = 11 \cdot 2 = 9 \in H$. $\quad$ $6 \cdot 11 = 1 \pmod{13}$
- $7H \neq 4H \iff 4^{-1} \cdot 7 = 10 \cdot 7 = 5 \notin H$. $\quad$ $4 \cdot 10 = 1 \pmod{13}$