# Abstract Algebra
## Day 3 Class Work

1. Consider the statement:

   Let $a,\ b,\ c \in \mathbb{Z}$. If $a \mid b$ and $b \mid c$, then $a \mid c$.

   (a) Create a few examples to illustrate this statement.

   (b) Write the first line and the last line of the proof.

   (c) Go ahead and prove the statement.

**GCD theorem:** Let $a,\ b \in \mathbb{Z}$. If $\gcd(a, b) = 1$, then there exist $x,\ y \in \mathbb{Z}$ with $ax + by = 1$.

← You'll be using this theorem often today.

2. Here's the *converse* of the GCD theorem:

   Let $a,\ b \in \mathbb{Z}$. If there exist $x,\ y \in \mathbb{Z}$ with $ax + by = 1$, then $\gcd(a, b) = 1$.

   **Recall:** The *converse* is obtained by swapping the if-part and the then-part.

   **The goal of this problem is to prove the above converse.**

   (a) What is the *hypothesis* of the converse? (i.e., what can we *assume* in the proof?)

   (b) What is the *conclusion* of the converse? (i.e., what must we *show* in the proof?)

   (c) Write the first line (or two) and the last line of the proof.

   (d) Complete the proof. Here are some hints:

   - Let $d = \gcd(a, b)$ so that $d \mid a$ and $d \mid b$. ← You may assume $d > 0$, as gcd is always positive.
   - Then show that $d$ is a divisor of $ax + by$.
   - Now show that $d = 1$. (By the way, the only positive divisor of 1 is. . . ?)

3. Determine if each of the following is true. Explain your reasoning.

   (a) $17 \mid 0$. (b) $0 \mid 17$. (c) $0 \mid 0$.

   **Ans to (c):** True. (Why?)

4. Consider the statement:

   Let $a,\ b,\ c \in \mathbb{Z}$. If $a \mid (bc)$ and $\gcd(a, b) = 1$, then $a \mid c$.

   (a) Create a few concrete examples to convince yourselves that the statement is true.

   (b) Is the statement still true *without* the condition $\gcd(a, b) = 1$? Why or why not? **Ans:** No. (Why not?)

   (c) Write the first line (or two) and the last line of the proof.

   (d) Go ahead and prove the statement.

   **Hint:** Use the GCD theorem to *translate* $\gcd(a, b) = 1$ into something more usable. ← i.e., $ax + by = 1$.

5. Proceed as in Problem #4 with this statement:

   Let $a,\ b,\ c \in \mathbb{Z}$. If $a \mid c,\ b \mid c,$ and $\gcd(a, b) = 1$, then $(ab) \mid c$.

6. Find *all* integer solutions to $5x + 8y = 1$. How do you know that you've found them all?

7. **(Some Food for Thought)** Prove the GCD theorem.