Abstract Algebra
Day 2 Class Work Solutions

1. List the elements of the following subsets of $\mathbb{Z}$.

   (a) $\{n \in \mathbb{Z} \mid 0 < n < 25 \text{ and } n \text{ is a multiple of } 3\}$

   Solution. $\{3, 6, 9, 12, 15, 18, 21, 24\}$.

   (b) $\{n \in \mathbb{Z} \mid 0 < 4n - 1 < 20\}$

   Hint for (b): It's *not* $\{3, 7, 11, 15, 19\}$.

   Solution. $\{1, 2, 3, 4, 5\}$.

2. Here's another subset of $\mathbb{Z}$:

$$3\mathbb{Z} = \{n \in \mathbb{Z} \mid n = 3k \text{ where } k \in \mathbb{Z}\}.$$

   (a) List a few elements of the set $3\mathbb{Z}$.

   ← There are infinitely many.

   Solution. The set $3\mathbb{Z}$ contains all integer multiples of 3, i.e.,

$$3\mathbb{Z} = \{\ldots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \ldots\}.$$

   (b) Choose two elements of $3\mathbb{Z}$ and add them together. Is the sum still in $3\mathbb{Z}$? Do this with a few pairs of elements in $3\mathbb{Z}$.

   Solution. Yes, the sum of any two elements of $3\mathbb{Z}$ is contained in $3\mathbb{Z}$. For instance, $6, 12 \in 3\mathbb{Z}$ and $6 + 12 = 18 \in 3\mathbb{Z}$.

   (c) **Prove:** If $m, n \in 3\mathbb{Z}$, then $m + n \in 3\mathbb{Z}$.

   ← If you're stuck, take a look at the next problem.

   Note: In other words, the set $3\mathbb{Z}$ is *closed under addition*.

   PROOF. Assume $m, n \in 3\mathbb{Z}$. Then $m = 3k$ and $n = 3j$ where $k, j \in \mathbb{Z}$. We have

$$m + n = 3k + 3j = 3(k + j) \in 3\mathbb{Z},$$

   since $k + j \in \mathbb{Z}$. Thus, $m + n \in 3\mathbb{Z}$. ■

3. Elizabeth and Anita wrote the following proof in Problem #2(c):

   Proof : Assume $m, n \in 3\mathbb{Z}$.
   Then $m = 3k$ and $n = 3j$ where $k, j \in \mathbb{Z}$.
   Thus, $m + n = 3(k + j) = 3k + 3j \in 3\mathbb{Z}$, since $k + j \in \mathbb{Z}$.
   Hence, $m + n \in 3\mathbb{Z}$.

   There is a (subtle) logical error in their proof. Find it and fix it.

   Solution. The equation $m + n = 3(k + j) = 3k + 3j \in 3\mathbb{Z}$ should be written as

$$m + n = 3k + 3j = 3(k + j) \in 3\mathbb{Z},$$

   to convey the correct logical order (i.e., substitute $3k$ and $3j$ first, then factor the 3).

4. Here's yet another subset of $\mathbb{Z}$:

$$C = \{n \in \mathbb{Z} \mid n \in 2\mathbb{Z} \text{ and } n \in 3\mathbb{Z}\}.$$

   (a) List a few elements of the set C.

Solution. Set C contains 6, 12, 24, 30, ..., as well as their negatives (and also 0).

(b) The set C has a more familiar name. What is it? Can you *prove* your conjecture?

← If you can't prove it (yet), that's okay. Move onto Problem #5 for now.

Solution. We conjecture that $C = 6\mathbb{Z}$.

5. Below, C refers to the set from Problem #4.

   (a) **Prove:** If $n \in 6\mathbb{Z}$, then $n \in C$. (**Hint:** If $n = 6k$, then $n = 2 \cdot \boxed{?}$ and $n = 3 \cdot \boxed{?}$.)

   Proof. Assume $n \in 6\mathbb{Z}$. Then $n = 6k$ with $k \in \mathbb{Z}$. We have $n = 6k = 2 \cdot (3k) \in 2\mathbb{Z}$ and $n = 6k = 3 \cdot (2k) \in 3\mathbb{Z}$. Thus, $n \in 2\mathbb{Z}$ and $n \in 3\mathbb{Z}$, so that $n \in C$. ■

   (b) **Prove:** If $n \in C$, then $n \in 6\mathbb{Z}$.

   (c) Anita says, "Parts (a) and (b) together prove that $C = 6\mathbb{Z}$." What might she mean?

   Solution. Part (a) proves that $6\mathbb{Z} \subseteq C$, i.e., $6\mathbb{Z}$ is a subset of C. Part (b) proves that $C \subseteq 6\mathbb{Z}$, i.e., C is a subset of $6\mathbb{Z}$. Taken together, those two relationships imply that that the sets C and $6\mathbb{Z}$ are equal.

6. Consider the set $\mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z},\, n \neq 0 \right\}$.

   (a) Write down a few elements of this set.

   (b) Find the sum $\frac{2}{5} + \frac{7}{3}$. (Be careful, it's *not* $\frac{9}{8}$.) Is this sum still in $\mathbb{Q}$?

   Ans to (b): $\frac{41}{15}$.

   (c) Choose two elements of $\mathbb{Q}$ and add them together. Is the sum still in $\mathbb{Q}$?

   (d) Repeat part (c) with a few pairs of elements in $\mathbb{Q}$.

   (e) **Prove:** If x and y are in $\mathbb{Q}$, then $x + y$ is in $\mathbb{Q}$.

   Solution. $\mathbb{Q}$ denotes the set of *rational numbers*. Here are some elements of $\mathbb{Q}$:

$$\frac{1}{2}, \frac{2}{5}, \frac{-7}{4}, \frac{5}{1}, \frac{7}{3}, \frac{18}{25}, \frac{31}{-178}, \ldots$$

   Adding, for example, $\frac{2}{5}$ and $\frac{7}{3}$, we obtain

$$\frac{2}{5} + \frac{7}{3} = \frac{6}{15} + \frac{35}{15} = \frac{6+35}{15} = \frac{41}{15},$$

   which is an element of $\mathbb{Q}$, because $41, 15 \in \mathbb{Z}$ and $15 \neq 0$.

7. Let S and T be sets. Define their *intersection*, denoted $S \cap T$, to be the set containing elements that are in both S and T.

   (a) Elizabeth says,

   > "In Problem #4, I conjectured that $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$."

   What might she mean?

   (b) Find a value of m such that $4\mathbb{Z} \cap 5\mathbb{Z} = m\mathbb{Z}$.

   (c) Find a value of m such that $7\mathbb{Z} \cap 10\mathbb{Z} = m\mathbb{Z}$.

   (d) Find a value of m such that $10\mathbb{Z} \cap 15\mathbb{Z} = m\mathbb{Z}$.

   (e) Any conjectures? Can you **prove** your conjecture?