Abstract Algebra
Day 2 Class Work

1. List the elements of the following subsets of $\mathbb{Z}$.
   (a) $\{n \in \mathbb{Z} \mid 0 < n < 25 \text{ and } n \text{ is a multiple of } 3\}$
   (b) $\{n \in \mathbb{Z} \mid 0 < 4n - 1 < 20\}$

   Hint for (b): It's *not* $\{3, 7, 11, 15, 19\}$.

2. Here's another subset of $\mathbb{Z}$:
   $$3\mathbb{Z} = \{n \in \mathbb{Z} \mid n = 3k \text{ where } k \in \mathbb{Z}\}.$$
   (a) List a few elements of the set $3\mathbb{Z}$. ← There are infinitely many.
   (b) Choose two elements of $3\mathbb{Z}$ and add them together. Is the sum still in $3\mathbb{Z}$? Do this with a few pairs of elements in $3\mathbb{Z}$.
   (c) **Prove:** If $m, n \in 3\mathbb{Z}$, then $m + n \in 3\mathbb{Z}$. ← If you're stuck, take a look at the next problem.

   Note: In other words, the set $3\mathbb{Z}$ is *closed under addition.*

3. Elizabeth and Anita wrote the following proof in Problem #2(c):

   Proof : Assume $m, n \in 3\mathbb{Z}$.
   Then $m = 3k$ and $n = 3j$ where $k, j \in \mathbb{Z}$.
   Thus, $m + n = 3(k + j) = 3k + 3j \in 3\mathbb{Z}$, since $k + j \in \mathbb{Z}$.
   Hence, $m + n \in 3\mathbb{Z}$.

   There is a (subtle) logical error in their proof. Find it and fix it.

4. Here's yet another subset of $\mathbb{Z}$:
   $$C = \{n \in \mathbb{Z} \mid n \in 2\mathbb{Z} \text{ and } n \in 3\mathbb{Z}\}.$$
   (a) List a few elements of the set C.
   (b) The set C has a more familiar name. What is it? Can you *prove* your conjecture? ← If you can't prove it (yet), that's okay. Move onto Problem #5 for now.

5. Below, C refers to the set from Problem #4.
   (a) **Prove:** If $n \in 6\mathbb{Z}$, then $n \in C$. (**Hint:** If $n = 6k$, then $n = 2 \cdot \boxed{?}$ and $n = 3 \cdot \boxed{?}$.)
   (b) **Prove:** If $n \in C$, then $n \in 6\mathbb{Z}$.
   (c) Anita says, "Parts (a) and (b) together prove that $C = 6\mathbb{Z}$." What might she mean?

6. Consider the set $\mathbb{Q} = \left\{\frac{m}{n} \mid m, n \in \mathbb{Z},\, n \neq 0\right\}$.
   (a) Write down a few elements of this set.
   (b) Find the sum $\frac{2}{5} + \frac{7}{3}$. (Be careful, it's *not* $\frac{9}{8}$.) Is this sum still in $\mathbb{Q}$? **Ans to (b):** $\frac{41}{15}$.
   (c) Choose two elements of $\mathbb{Q}$ and add them together. Is the sum still in $\mathbb{Q}$?
   (d) Repeat part (c) with a few pairs of elements in $\mathbb{Q}$.

(e) **Prove:** If x and y are in $\mathbb{Q}$, then $x + y$ is in $\mathbb{Q}$.

7. Let S and T be sets. Define their *intersection*, denoted $S \cap T$, to be the set containing elements that are in both S and T.

   (a) Elizabeth says,

   > "In Problem #4, I conjectured that $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$."

   What might she mean?

   (b) Find a value of m such that $4\mathbb{Z} \cap 5\mathbb{Z} = m\mathbb{Z}$.

   (c) Find a value of m such that $7\mathbb{Z} \cap 10\mathbb{Z} = m\mathbb{Z}$.

   (d) Find a value of m such that $10\mathbb{Z} \cap 15\mathbb{Z} = m\mathbb{Z}$.

   (e) Any conjectures? Can you **prove** your conjecture?