Discuss in your group:

What is the smallest group (in terms of the number of elements) you can think of? Be sure to specify the set and the operation.

Examples:

- $\{0\}$ under addition.
- $\{1\}$ under multiplication.
- $\{\varepsilon\}$ under $\circ$ (composition).

Table for $\{0\}$

$+$	0
0	0

Group properties:

1. $\{0\}$ is closed under addition.
2. Associative law, i.e., $(0+0)+0 = 0+(0+0)$.
3. $\{0\}$ contains the *additive* identity 0.
4. Elements in $\{0\}$ have *additive* inverses.

Problem #2: Groups with three elements

(a) Notice how these group tables are essentially the same.

Table for $\mathbb{Z}_3$:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Table for $\{\varepsilon, r_{120}, r_{240}\}$:

$\circ$	ε	r_{120}	r_{240}
ε	ε	r_{120}	r_{240}
r_{120}	r_{120}	r_{240}	ε
r_{240}	r_{240}	ε	r_{120}

Table for $\{\varepsilon, \sigma, \tau\}$:

$\circ$	ε	σ	τ
ε	ε	σ	τ
σ	σ	τ	ε
τ	τ	ε	σ

(b) Let $G = \{\varepsilon, a, b\}$ be a three-element group. Then...

	ε	a	b
ε	ε	a	b
a	a	b	ε
b	b	ε	a

The Sudoku property

To derive the group table for $G = \{\varepsilon, a, b\}$, this group property helps:

* Each row/column of the table of a group G contains every element of G exactly once.

	ε	a	b
ε	ε	a	b
a	a		ε
b	b		

* $ab = \varepsilon$

Sudoku property: Let G be a group. In each row or column of its group table, every element of G shows up exactly once.

Proof outline: Let $G = \{\varepsilon, a, b, \ldots, g, \ldots\}$, possibly infinite.

	ε	a	b	$\cdots$	g	$\cdots$	$\cdots$
ε							
a							
$\vdots$							
g	g	ga	gb	$\cdots$	gg	$\cdots$	$\cdots$
$\vdots$							

$\longleftarrow$ row g

Focusing on row g, we must show that:

(1) The elements in this row are all different. at most once.

(2) Every $x \in G$ appears in this row. at least once.

	ε	a	$\neq$	b	$\cdots$	g	$\cdots$	$\cdots$	
ε									
a			?						
$\vdots$									
g	g	ga	$\neq$	gb	$\cdots$	gg	$\cdots$	$\cdots$	$\longleftarrow$ row g
$\vdots$									

(1) The elements in this row are all different.

- This amounts to showing: If $a \neq b$, then $ga \neq gb$.
- Its contrapositive is: If $\cancel{g}a = \cancel{g}b$, then $a = b$ (left cancellation).

$= g^{-1} \cdot x$

	ε	a	b	$\cdots$	g	$\cdots$	$??$	$\cdots$	
ε									
a									
$\vdots$									
g	g	ga	gb	$\cdots$	gg	$\cdots$	x	$\cdots$	$\longleftarrow$ row g
$\vdots$									

(2) Every $x \in G$ appears in this row.

- Given $x \in G$, we need an element $?? \in G$ such that $g \cdot ?? = x$.
- With $?? = g^{-1}x$, we have $g \cdot (g^{-1}x) = (gg^{-1})x = \varepsilon x = x$.