### Discuss in your group:

What is the smallest group (in terms of the number of elements) you can think of? Be sure to specify the set and the operation.

## **Examples:**

- {0} under addition.
- {1} under multiplication.
- $\{\varepsilon\}$  under  $\circ$  (composition).

Table 
$$+ 0$$
 for  $\{0\}$   $0$   $0$ 

#### Group properties:

- 1. {0} is closed under addition.
- 2. Associative law, i.e., (0+0)+0=0+(0+0).
- 3.  $\{0\}$  contains the *additive* identity 0.
- 4. Elements in  $\{0\}$  have additive inverses.

## **Problem #2:** Groups with three elements

(a) Notice how these group tables are essentially the same.

Table for  $\mathbb{Z}_3$ :

Table for  $\{\varepsilon, r_{120}, r_{240}\}$ :

0	$ \varepsilon $	$r_{120}$	$r_{240}$
ε	$\varepsilon$	$r_{120}$	$r_{240}$
$r_{120}$	$r_{120}$	$r_{240}$	$\varepsilon$
$r_{240}$	$r_{240}$	$\varepsilon$	$r_{120}$

Table for  $\{\varepsilon, \sigma, \tau\}$ :

0	$\varepsilon$	$\sigma$	$\tau$
$\varepsilon$	ε	$\sigma$	$\tau$
$\sigma$	$\sigma$	$\tau$	ω
au	au	$\omega$	σ

(b) Let  $G = \{\varepsilon, a, b\}$  be a three-element group. Then...

	$\varepsilon$	a	b
$\varepsilon$	3	a	6
a	a	Ь	3
b	6	8	0

# The Sudoku property

To derive the group table for  $G = \{\varepsilon, a, b\}$ , this group property helps:

 $\star$  Each row/column of the table of a group G contains every element of G exactly once.

	arepsilon	a	b	_	
$\varepsilon$	$\varepsilon$	a	b	-	
a	a		ج	*	$ab = \varepsilon$
b	b			_	

**Sudoku property:** Let G be a group. In each row or column of its group table, every element of G shows up exactly once.

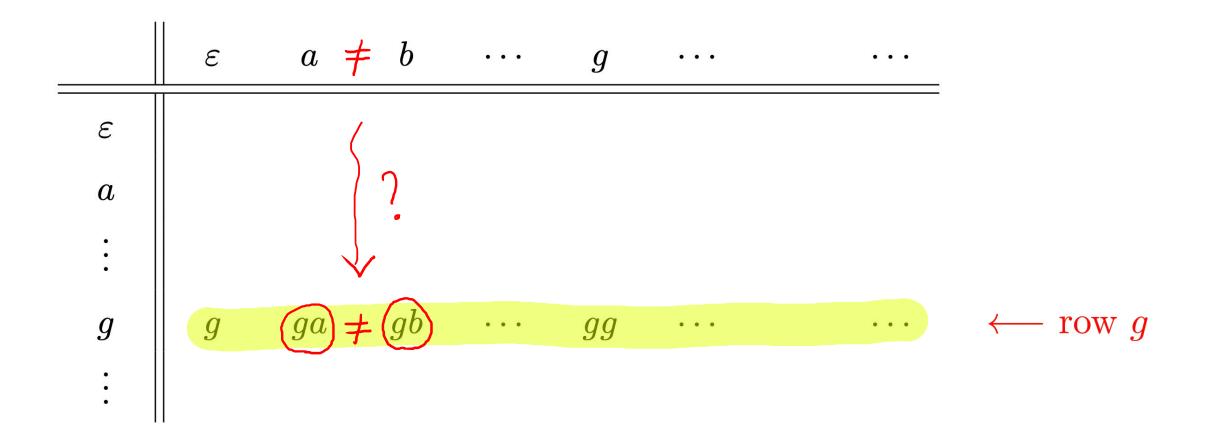
**Proof outline:** Let  $G = \{\varepsilon, a, b, \ldots, g, \ldots\}$ , possibly infinite.

	arepsilon	a	b	• • •	g		•••	
arepsilon								
a								
:								
g	g	ga	gb		gg	$)$ $\cdot \cdot \cdot \cdot )$ $)$		$\leftarrow$ row $g$
÷								

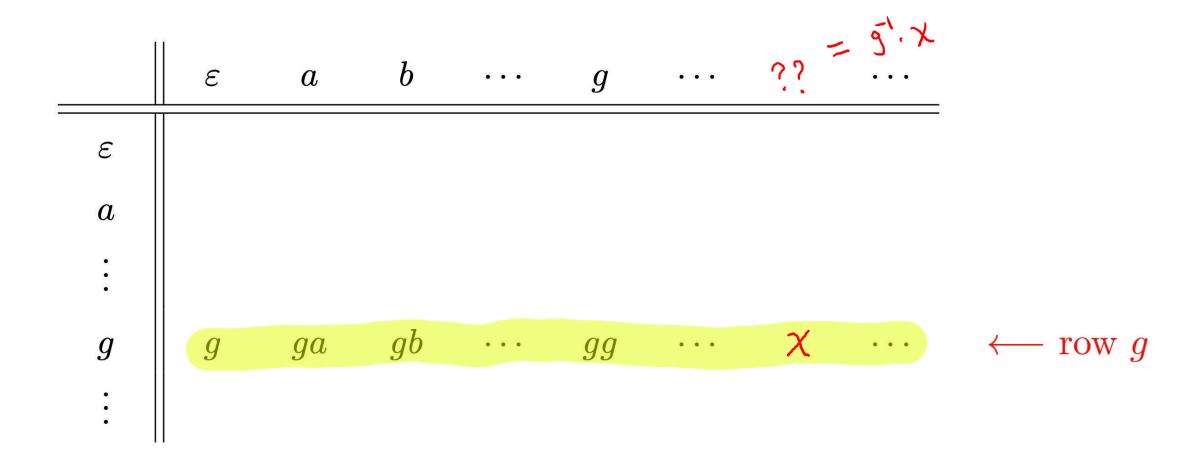
Focusing on row g, we must show that:

- (1) The elements in this row are all different. at most on ce.
- (2) Every  $x \in G$  appears in this row. at least once.





- (1) The elements in this row are all different.
  - This amounts to showing: If  $a \neq b$ , then  $ga \neq gb$ .
  - Its contrapositive is: If ga = gb, then a = b (left cancellation).



- (2) Every  $x \in G$  appears in this row.
  - Given  $x \in G$ , we need an element  $?? \in G$  such that  $g \cdot ?? = x$ .
  - With  $?? = g^{-1}x$ , we have  $g \cdot (g^{-1}x) = (gg^{-1})x = \varepsilon x = x$ .

