Abstract Algebra
Day 9 Class Work Solutions

1. Consider the groups: $\{1, -1\}$ under multiplication, and $\mathbb{Z}_2 = \{0, 1\}$ under addition. ← For the first group, treat 1 and -1 as real numbers.

   (a) Construct the group table for each group.

   Solution.

   Table for $\{1, -1\}$:

$*$	1	-1
1	1	-1
-1	-1	1

   Table for $\mathbb{Z}_2$:

$+$	0	1
0	0	1
1	1	0

   (b) Anita says the two tables in part (a) are essentially the same. What might she mean?

   Solution. They both look like the table in part (c) below.

   (c) Let $G = \{\varepsilon, g\}$ be a two-element group. Complete its group table: ← Use the multiplicative notation for a general group.

   Solution.

	ε	g
ε	ε	g
g	g	ε

   (d) Explain why all groups with two elements are essentially the same.

   Solution. Every group with two elements will have the identity ε and the non-identity element g. Their tables must look like the one in part (c). Since ε is the identity, we have $\varepsilon\varepsilon = \varepsilon$, $\varepsilon g = g$, and $g\varepsilon = g$. For the remaining entry gg, we note that either $gg = \varepsilon$ or $gg = g$ since the group is closed. Suppose for contradiction that $gg = g$. Then we have $gg = g = g\varepsilon$, and thus we obtain $g = \varepsilon$ by left cancellation. This is a contradiction, so we must have $gg = \varepsilon$.

2. Consider the groups:

   - $\mathbb{Z}_3 = \{0, 1, 2\}$ under addition
   - $\{\varepsilon, r_{120}, r_{240}\}$ under the operation $\circ$ in D_3.
   - $\{\varepsilon, \sigma, \tau\}$ under the operation $\circ$ in S_3, where $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

   (a) Construct the group table for each group. Verify that they're essentially the same.

   Solution. All three tables look like the table in part (b).

   Table for $\mathbb{Z}_3$:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

   Table for $\{\varepsilon, r_{120}, r_{240}\}$:

$\circ$	ε	r_{120}	r_{240}
ε	ε	r_{120}	r_{240}
r_{120}	r_{120}	r_{240}	ε
r_{240}	r_{240}	ε	r_{120}

   Table for $\{\varepsilon, \sigma, \tau\}$:

$\circ$	ε	σ	τ
ε	ε	σ	τ
σ	σ	τ	ε
τ	τ	ε	σ

   (b) Let $G = \{\varepsilon, a, b\}$ be a three-element group. Complete its group table: ← We have $ab = \varepsilon$. Why?

   Solution.

	ε	a	b
ε	ε	a	b
a	a	b	ε
b	b	ε	a

(c) Explain why all groups with three elements are essentially the same.

Solution. Let $G = \{\varepsilon, a, b\}$ be a three element group, with identity ε and two non-identity elements a and b. We will build its table in phases. In the first phase, we can fill in the first row and column of the table, since $\varepsilon x = x$ and $x\varepsilon = x$ for all $x \in G$. In the second phase, we note that $ab = \varepsilon$. Here, we use a property of the group that we've seen before, namely: in each row or column of the table, each element of G shows up exactly once. Then, ab cannot equal a, since a already showed up in the second row; and ab cannot equal b, since b already showed up in the third column. Thus, ab must equal ε. We use the same property to complete the table.

Phase 1:

	ε	a	b
ε	ε	a	b
a	a		
b	b		

Phase 2:

	ε	a	b
ε	ε	a	b
a	a		ε
b	b		

Complete table:

	ε	a	b
ε	ε	a	b
a	a	b	ε
b	b	ε	a

3. Let's look at these four-element groups:

- $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ under addition
- $\{1, i, -1, -i\}$ under multiplication.
- $U_8 = \{1, 3, 5, 7\}$ under multiplication

← i is the complex number $i = \sqrt{-1}$, where $i^2 = -1$.

(a) Construct the group table for each group.

Solution.

Table for $\mathbb{Z}_4$:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Table for $\{1, i, -1, -i\}$:

$*$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

Table for U_8:

$*$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

(b) One of these groups is not like the others. Which one? How do you know?

← . . . says Big Bird.

Solution. Both $\mathbb{Z}_4$ and $\{1, i, -1, -i\}$ have two self inverses; and their tables look the same. Meanwhile, every element of U_8 is a self inverse and its table looks different from the other two groups.

4. (a) Come with another group with four elements. Specify the set and the operation.

(b) Does your group in part (a) resemble $\mathbb{Z}_4$ or U_8? How do you know?

(c) Find another four-element group that's *not* like the one you found in part (a).

5. **(Some Food for Thought)** Explain why all groups with four elements must resemble either of the two types that you saw in Problem #3.

6. **(More Food for Thought)** What about groups with 5 elements? 6 elements? 7 elements?

7. **Prove:** Let G be a group. In each row or column of its group table, every element of G shows up exactly once.

← i.e., Sudoku property.

Solution. See Section 9.4 in the textbook for the proof.