© 2022 Ryota Matsuura

Abstract Algebra
Day 9 Class Work

1. Consider the groups: $\{1, -1\}$ under multiplication, and $\mathbb{Z}_2 = \{0, 1\}$ under addition. ← For the first group, treat 1 and -1 as real numbers.
   (a) Construct the group table for each group.
   (b) Anita says the two tables in part (a) are essentially the same. What might she mean?
   (c) Let $G = \{\varepsilon, g\}$ be a two-element group. Complete its group table: ← Use the multiplicative notation for a general group.

	ε	g
ε		
g		

   (d) Explain why all groups with two elements are essentially the same.

2. Consider the groups:
   - $\mathbb{Z}_3 = \{0, 1, 2\}$ under addition
   - $\{\varepsilon, r_{120}, r_{240}\}$ under the operation $\circ$ in D_3.
   - $\{\varepsilon, \sigma, \tau\}$ under the operation $\circ$ in S_3, where $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

   (a) Construct the group table for each group. Verify that they're essentially the same.
   (b) Let $G = \{\varepsilon, a, b\}$ be a three-element group. Complete its group table: ← We have $ab = \varepsilon$. Why?

	ε	a	b
ε			
a			
b			

   (c) Explain why all groups with three elements are essentially the same.

3. Let's look at these four-element groups:
   - $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ under addition
   - $\{1, i, -1, -i\}$ under multiplication. ← i is the complex number $i = \sqrt{-1}$, where $i^2 = -1$.
   - $U_8 = \{1, 3, 5, 7\}$ under multiplication

   (a) Construct the group table for each group.
   (b) One of these groups is not like the others. Which one? How do you know? ← . . . says Big Bird.

4. (a) Come with another group with four elements. Specify the set and the operation.
   (b) Does your group in part (a) resemble $\mathbb{Z}_4$ or U_8? How do you know?
   (c) Find another four-element group that's *not* like the one you found in part (a).

5. **(Some Food for Thought)** Explain why all groups with four elements must resemble either of the two types that you saw in Problem #3.

6. **(More Food for Thought)** What about groups with 5 elements? 6 elements? 7 elements?

7. **Prove:** Let G be a group. In each row or column of its group table, every element of G shows up exactly once. ← i.e., Sudoku property.