Abstract Algebra
Day 8 Class Work

1. Come up with as many examples of groups as possible. Be sure to...
   - Specify both the set (e.g., S_3) and the operation (e.g., $\circ$).
   - Check the group properties 1 through 4.
   - Determine if the group is commutative or non-commutative.

2. Here are some examples of groups under *addition*: $\mathbb{Z}$, $\mathbb{Z}_{10}$, $\mathbb{Z}_7$. Each of these sets is also closed under multiplication. Are they *groups* under multiplication? Why or why not?

   Ans: No. (Why not?)

3. Recall that $U_{10} = \{a \in \mathbb{Z}_{10} \mid a \text{ has a } \textit{multiplicative} \text{ inverse in } \mathbb{Z}_{10}\}$.

   Note: We must be specific about the type of inverse (multiplicative or additive).

   (a) Find the elements of U_{10}. (**Hint:** There are four of them.)

   (b) Construct a multiplication table for U_{10}.

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   (c) Use the table to check the group properties for U_{10}.
   Note: You may simply assume that multiplication is associative.

   (d) Is U_{10} commutative or non-commutative?

4. With σ, $\tau \in S_3$, we've seen that $(\sigma\tau)^{-1} \neq \sigma^{-1}\tau^{-1}$, but instead $(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}$.

   (a) Compute $(\sigma\tau)(\tau^{-1}\sigma^{-1})$ and $(\tau^{-1}\sigma^{-1})(\sigma\tau)$.

   Hint: Use the associative law to regroup elements.

   (b) Using your results in part (a), explain why $(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}$ is *always* true.

   (c) Why might this be called the "socks-shoes" property?

5. In S_3, consider again
$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

   (a) Elizabeth says, "σ^{-5} is the same as $(\sigma^{-1})^5$." Anita claims that $\sigma^{-5} = (\sigma^5)^{-1}$. With whom do you agree?

   (b) Compute $(\sigma^{-1})^5$ and $(\sigma^5)^{-1}$, and verify that they're equal.

   ← Therefore, we can unambiguously say σ^{-5}.

6. **Theorem (left cancellation):** Let a, b, c be elements of a group. If $ab = ac$, then $b = c$.

   (a) In $\mathbb{Z}_{10}$, we have $2 \cdot 6 = 2 \cdot 1$, even though $6 \neq 1$. Explain why this does *not* violate the (left) cancellation law.

   Hint: See Problem #2.

   (b) Define σ, τ, $\mu \in S_3$, respectively, by
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$
   Compute the products $\sigma\tau$ and $\mu\sigma$ and verify that they're equal.

   (c) Anita says, "We have $\sigma\tau = \mu\sigma$. If we cancel σ from both sides, we get $\tau = \mu$. How's that possible?" How would you respond to her?

7. Prove the left cancellation law, stated in Problem #6. In your proof, be sure to specify the group properties that are used.

Note: Right cancellation is defined/proved similarly.

8. In Problem #3, we saw that U_{10} is a group under multiplication.

   (a) What about U_{35}? Is it a group under multiplication? How do you know?

   ← Constructing a table is *not* recommended...

   (b) More generally, explain why U_m is a group under multiplication.

9. (a) Consider $U_8 = \{1, 3, 5, 7\}$. Verify that $a^2 = 1$ for all $a \in U_8$.

   ← U_8 is a group under multiplication.

   (b) Is U_8 commutative or non-commutative?

   (c) Repeat parts (a) and (b) with the group U_{12}. (First find the elements of U_{12}.)

10. Consider the subset $C(h) = \{\varepsilon, r_{180}, h, v\}$ of D_4. In Day 5 Class Work, we verified that $C(h)$ is a group under composition.

    Recall: $C(h)$ is called the *centralizer* of h in D_4.

    (a) Now verify that $\alpha^2 = \varepsilon$ for all $\alpha \in C(h)$.

    (b) Is $C(h)$ commutative or non-commutative?

    Ans: Commutative.

11. (a) **Prove:** Let G be a group. If $g^2 = \varepsilon$ for all $g \in G$, then G is commutative.

    (b) Is the converse of the statement in part (a) true? If it's true, prove it. If it's false, give a counterexample.