Abstract Algebra
Day 7 Class Work Solutions

1. Let $\alpha,\ \beta \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$.

   (a) We've seen that $\alpha + \beta = \left[\begin{smallmatrix} 6 & 8 \\ 0 & 2 \end{smallmatrix}\right]$. This time, compute $\beta + \alpha$.

   Solution. $\beta + \alpha = \left[\begin{smallmatrix} 6 & 8 \\ 0 & 2 \end{smallmatrix}\right]$.

   (b) We've seen that $\alpha \cdot \beta = \left[\begin{smallmatrix} 9 & 2 \\ 3 & 0 \end{smallmatrix}\right]$. This time, compute $\beta \cdot \alpha$.

   Ans: $\beta \cdot \alpha = \left[\begin{smallmatrix} 3 & 4 \\ 1 & 6 \end{smallmatrix}\right]$.

   Solution. $\beta \cdot \alpha = \left[\begin{smallmatrix} 3 & 4 \\ 1 & 6 \end{smallmatrix}\right]$.

   (c) Is addition in $M(\mathbb{Z}_{10})$ commutative? Why or why not?

   Solution. Yes. We'll leave the explanation to you!

   (d) Is multiplication in $M(\mathbb{Z}_{10})$ commutative? Why or why not?

   Solution. No. We saw in part (b) that $\alpha \cdot \beta \neq \beta \cdot \alpha$.

2. Again, let $\alpha,\ \beta \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$.

   (a) Compute each of these sums: $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \alpha,\ \ \alpha + \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right],\ \ \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \beta,\ \ \beta + \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

   Solution. We have $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \alpha = \alpha,\ \ \alpha + \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] = \alpha,\ \ \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \beta = \beta,\ \ \beta + \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] = \beta$.

   (b) Explain why $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is called the *additive identity* element of $M(\mathbb{Z}_{10})$.

   ← i.e, $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \in M(\mathbb{Z}_{10})$ behaves like $0 \in \mathbb{Z}$.

   Solution. Adding to $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ to any matrix keeps that matrix unchanged.

3. Again, let $\alpha,\ \beta \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$.

   (a) Find $\tau \in M(\mathbb{Z}_{10})$ such that $\alpha + \tau = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

   Solution. $\tau = \left[\begin{smallmatrix} 9 & 8 \\ 7 & 6 \end{smallmatrix}\right]$.

   (b) Find $\tau \in M(\mathbb{Z}_{10})$ such that $\tau + \beta = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

   Solution. $\tau = \left[\begin{smallmatrix} 5 & 4 \\ 3 & 2 \end{smallmatrix}\right]$.

   (c) Given $\sigma \in M(\mathbb{Z}_{10})$, describe how you'd find $\tau \in M(\mathbb{Z}_{10})$ such that

   $$\sigma + \tau = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \ \text{ and } \ \tau + \sigma = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right].$$

   Solution. Given $\sigma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M(\mathbb{Z}_{10})$, let $\tau = \left[\begin{smallmatrix} 10-a & 10-b \\ 10-c & 10-d \end{smallmatrix}\right]$.

4. Verify that $M(\mathbb{Z}_{10})$ is a group under addition.

   ← Assume that addition in $M(\mathbb{Z}_{10})$ is associative.

   Solution.

   1. $M(\mathbb{Z}_{10})$ is closed under addition. For any pair of matrices $\alpha,\ \beta \in M(\mathbb{Z}_{10})$, their sum $\alpha + \beta$ is also in $M(\mathbb{Z}_{10})$. After all, addition in $M(\mathbb{Z}_{10})$ is based on addition in $\mathbb{Z}_{10}$.
   2. We'll assume (for now) that addition in $M(\mathbb{Z}_{10})$ is associative, i.e., for all α, β, $\gamma \in M(\mathbb{Z}_{10})$, we have $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
   3. $M(\mathbb{Z}_{10})$ has an additive identity element $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. See Problem #2.
   4. Every element in $M(\mathbb{Z}_{10})$ has an additive inverse. See Problem #3.

5. Let $\varepsilon \in M(\mathbb{Z}_{10})$ where $\varepsilon = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

   (a) With $\alpha = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$, compute the products $\varepsilon \cdot \alpha$ and $\alpha \cdot \varepsilon$.

   Solution. We have $\varepsilon \cdot \alpha = \alpha$ and $\alpha \cdot \varepsilon = \alpha$.

(b) **Optional:** With $\beta = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$, compute the products $\varepsilon \cdot \beta$ and $\beta \cdot \varepsilon$.

Solution. We have $\varepsilon \cdot \beta = \beta$ and $\beta \cdot \varepsilon = \beta$.

(c) Explain why ε is called the *multiplicative identity* element of $M(\mathbb{Z}_{10})$.

← $\varepsilon \in M(\mathbb{Z}_{10})$ behaves like which element in $\mathbb{Z}$?

Solution. Multiplying ε with any matrix keeps that matrix unchanged. Therefore, $\varepsilon \in M(\mathbb{Z}_{10})$ behaves like $1 \in \mathbb{Z}$.

6. Let σ, $\tau \in M(\mathbb{Z}_{10})$ where $\sigma = \left[\begin{smallmatrix} 2 & 1 \\ 7 & 4 \end{smallmatrix}\right]$ and $\tau = \left[\begin{smallmatrix} 4 & 9 \\ 3 & 2 \end{smallmatrix}\right]$.

(a) Compute the products $\sigma \cdot \tau$ and $\tau \cdot \sigma$.

Solution. We have $\sigma \cdot \tau = \varepsilon$ and $\tau \cdot \sigma = \varepsilon$, where $\varepsilon = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

(b) How are the elements σ and τ related?

Solution. They are multiplicative inverses of each other.

7. Determine if it's possible to find a matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in $M(\mathbb{Z}_{10})$ such that $\left[\begin{smallmatrix} 2 & 3 \\ 0 & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \varepsilon$. If so, then find it. If not, then explain why not.

← Here, $\varepsilon = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

Solution. It is *not* possible. We have $\left[\begin{smallmatrix} 2 & 3 \\ 0 & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right]$, where $x = 2a + 3c$ and $y = 2b + 3d$. Then $\left[\begin{smallmatrix} 2 & 3 \\ 0 & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \neq \varepsilon$, because the bottom row of $\left[\begin{smallmatrix} 2 & 3 \\ 0 & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ will always contain 0 and 0. Thus, $\left[\begin{smallmatrix} 2 & 3 \\ 0 & 0 \end{smallmatrix}\right]$ does *not* have a multiplicative inverse in $M(\mathbb{Z}_{10})$.

8. Is $M(\mathbb{Z}_{10})$ is a group under multiplication? Why or why not?

Ans: It's not. (Why not?)

Solution. $M(\mathbb{Z}_{10})$ is *not* a group under multiplication. Even though $M(\mathbb{Z}_{10})$ is closed under multiplication, multiplication in $M(\mathbb{Z}_{10})$ is associative, and $M(\mathbb{Z}_{10})$ has a multiplicative identity element ε, we saw in Problem #7 that not every element in $M(\mathbb{Z}_{10})$ has a multiplicative inverse.

9. As in Problem #6, let σ, $\tau \in M(\mathbb{Z}_{10})$ where $\sigma = \left[\begin{smallmatrix} 2 & 1 \\ 7 & 4 \end{smallmatrix}\right]$ and $\tau = \left[\begin{smallmatrix} 4 & 9 \\ 3 & 2 \end{smallmatrix}\right]$. We saw that $\sigma \cdot \tau = \varepsilon$ and $\tau \cdot \sigma = \varepsilon$, so that σ and τ are *multiplicative inverses* of each other in $M(\mathbb{Z}_{10})$. Our friends are having the following conversation:

Elizabeth: If we were given just $\sigma = \left[\begin{smallmatrix} 2 & 1 \\ 7 & 4 \end{smallmatrix}\right]$, how could we have found τ on our own?

Anita: Well, the diagonal entries of τ are the same as σ, but swapped. The 2 and 4 in σ become 4 and 2 and τ. But how do we get the 9 and 3 in τ?

Elizabeth: Hmm, we have 1 and 7 in σ... I got it! *Negate* them to get the 9 and 3.

Now suppose $\alpha \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} 6 & 7 \\ 1 & 3 \end{smallmatrix}\right]$. Find its multiplicative inverse β and verify your answer by computing $\alpha \cdot \beta$ and $\beta \cdot \alpha$.

Ans: $\beta = \left[\begin{smallmatrix} 3 & 3 \\ 9 & 6 \end{smallmatrix}\right]$.

Solution. We have $\beta = \left[\begin{smallmatrix} 3 & 3 \\ 9 & 6 \end{smallmatrix}\right]$. We'll leave it to you to show that $\alpha \cdot \beta = \varepsilon$ and $\beta \cdot \alpha = \varepsilon$.

10. Let $\alpha \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} 2 & 1 \\ 5 & 4 \end{smallmatrix}\right]$.

(a) Explain why our friends might say that $\beta = \left[\begin{smallmatrix} 4 & 9 \\ 5 & 2 \end{smallmatrix}\right]$ is the multiplicative inverse of α.

Solution. We obtain β by swapping the diagonal entries of α (i.e., 2 and 4) and negating the other two entries (i.e., $-1 = 9$ and $-5 = 5$).

(b) Compute $\alpha \cdot \beta$. Were Elizabeth and Anita correct?

Ans: Unfortunately, no.

Solution. We have $\alpha \cdot \beta = \left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$, so that $\alpha \cdot \beta = 3 \cdot \varepsilon$. Since $\alpha \cdot \beta \neq \varepsilon$, we conclude that α and β are *not* multiplicative inverses of each other.

(c) Instead, let $\tau = 7 \cdot \beta$, i.e., multiply each entry of β by 7 (and reduce modulo 10). Now compute $\alpha \cdot \tau$ and $\tau \cdot \alpha$.

Solution. We have $\tau = \left[\begin{smallmatrix} 8 & 3 \\ 5 & 4 \end{smallmatrix}\right]$; and we find that $\alpha \cdot \tau = \varepsilon$ and $\tau \cdot \alpha = \varepsilon$.

11. Let $\alpha \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} 2 & 3 \\ 1 & 5 \end{smallmatrix}\right]$. Find the multiplicative inverse of α.

    Note: Be sure to verify your answer!

    Solution. We proceed as in Problem #10. We first obtain $\beta = \left[\begin{smallmatrix} 5 & -3 \\ -1 & 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 5 & 7 \\ 9 & 2 \end{smallmatrix}\right]$. We then find that $\alpha \cdot \beta = \left[\begin{smallmatrix} 7 & 0 \\ 0 & 7 \end{smallmatrix}\right] = 7 \cdot \varepsilon$. Noting that 3 and 7 are multiplicative inverses of each other in $\mathbb{Z}_{10}$, we let $\tau = 3 \cdot \beta = \left[\begin{smallmatrix} 5 & 1 \\ 7 & 6 \end{smallmatrix}\right]$. Then we have

    $$\alpha \cdot \tau = \alpha \cdot (3 \cdot \beta) = 3 \cdot (\alpha \cdot \beta) = 3 \cdot (7 \cdot \varepsilon) = (3 \cdot 7) \cdot \varepsilon = 1 \cdot \varepsilon = \varepsilon,$$

    so that $\alpha \cdot \tau = \varepsilon$. (We can also verify this directly by multiplying $\alpha = \left[\begin{smallmatrix} 2 & 3 \\ 1 & 5 \end{smallmatrix}\right]$ and $\tau = \left[\begin{smallmatrix} 5 & 1 \\ 7 & 6 \end{smallmatrix}\right]$.) We can similarly verify that $\tau \cdot \alpha = \varepsilon$. Thus, $\tau = \left[\begin{smallmatrix} 5 & 1 \\ 7 & 6 \end{smallmatrix}\right]$ is the multiplicative inverse of α.

12. Let $\alpha \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.

    (a) Describe how you'd find the multiplicative inverse of α.

    (b) Elizabeth asks, "We've seen matrices that don't have multiplicative inverses, like in Problem #7. How can we tell whether or not a matrix has a multiplicative inverse?"

    Solution. Let's proceed as in Problem #10. We first obtain $\beta = \left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$, i.e., swap the diagonal entries of α (i.e., a and d) and negate the other two entries (i.e., $-b$ and $-c$). We then find that $\alpha \cdot \beta = \Delta \cdot \varepsilon$, where $\Delta = ad - bc$ is in $\mathbb{Z}_{10}$. If Δ has a multiplicative inverse in $\mathbb{Z}_{10}$, then let $\tau = \Delta^{-1} \cdot \beta$. Then we have

    $$\alpha \cdot \tau = \alpha \cdot (\Delta^{-1} \cdot \beta) = \Delta^{-1} \cdot (\alpha \cdot \beta) = \Delta^{-1} \cdot (\Delta \cdot \varepsilon) = (\Delta^{-1} \cdot \Delta) \cdot \varepsilon = 1 \cdot \varepsilon = \varepsilon,$$

    so that $\alpha \cdot \tau = \varepsilon$. We can similarly verify that $\tau \cdot \alpha = \varepsilon$. Thus, if $\Delta = ad - bc$ has a multiplicative inverse in $\mathbb{Z}_{10}$, then $\tau = \Delta^{-1} \cdot \left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$ is the multiplicative inverse of α.

13. Explain why it's impossible to find a matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in $M(\mathbb{Z}_{10})$ such that $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \varepsilon$. ← Here, $\varepsilon = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

14. Repeat Problem #13 with the matrix $\left[\begin{smallmatrix} 2 & 1 \\ 5 & 5 \end{smallmatrix}\right]$.

Definition: Let $\alpha \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. The *determinant* of α, denoted $\det \alpha$, is given by $\det \alpha = ad - bc$. Note that $\det \alpha$ is a number in $\mathbb{Z}_{10}$.

15. (a) Let $\alpha, \beta \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$. Find $\det \alpha$ and $\det \beta$. **Ans:** $\det \alpha = 8$.

    Solution. We have $\det \alpha = 1 \cdot 4 - 2 \cdot 3 = 8$ and $\det \beta = 5 \cdot 8 - 6 \cdot 7 = 8$.

    (b) Find the determinant of each matrix in Problems #9, #10, and #11.

    Solution.

    - Problem #9: We have $\alpha = \left[\begin{smallmatrix} 6 & 7 \\ 1 & 3 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 3 & 3 \\ 9 & 6 \end{smallmatrix}\right]$ with $\det \alpha = 1$ and $\det \beta = 1$.
    - Problem #10: We have $\alpha = \left[\begin{smallmatrix} 2 & 1 \\ 5 & 4 \end{smallmatrix}\right]$, $\beta = \left[\begin{smallmatrix} 4 & 9 \\ 5 & 2 \end{smallmatrix}\right]$, and $\tau = \left[\begin{smallmatrix} 8 & 3 \\ 5 & 4 \end{smallmatrix}\right]$. Their determinants are $\det \alpha = 3$, $\det \beta = 3$, and $\det \tau = 7$.
    - Problem #11: We have $\alpha = \left[\begin{smallmatrix} 2 & 3 \\ 1 & 5 \end{smallmatrix}\right]$, $\beta = \left[\begin{smallmatrix} 5 & 7 \\ 9 & 2 \end{smallmatrix}\right]$, and $\tau = \left[\begin{smallmatrix} 5 & 1 \\ 7 & 6 \end{smallmatrix}\right]$. Their determinants are $\det \alpha = 7$, $\det \beta = 7$, and $\det \tau = 3$.

    (c) While you're at it, find the determinants of the matrices in Problems #13 and #14.

    Solution.

    - Problem #13: We have $\alpha = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right]$ with $\det \alpha = 0$.
    - Problem #14: We have $\alpha = \left[\begin{smallmatrix} 2 & 1 \\ 5 & 5 \end{smallmatrix}\right]$ with $\det \alpha = 5$.

    (d) What does the determinant of a matrix have to do with its multiplicative inverse?

    Solution. Let $\alpha \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, and let $\Delta = \det \alpha = ad - bc \in \mathbb{Z}_{10}$. We conjecture that if Δ has a multiplicative inverse in $\mathbb{Z}_{10}$, then α has a multiplicative inverse in $M(\mathbb{Z}_{10})$. Moreover, if Δ does *not* have a multiplicative inverse in $\mathbb{Z}_{10}$, then α does *not* have a multiplicative inverse in $M(\mathbb{Z}_{10})$.

16. Let α, $\beta \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} 2 & 1 \\ 5 & 4 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 3 & 4 \\ 1 & 2 \end{smallmatrix}\right]$.

    (a) Compute $\det \alpha$, $\det \beta$, and $\det(\alpha \cdot \beta)$. How are these determinants related?

    Note: For $\det(\alpha \cdot \beta)$, you must first find the product $\alpha \cdot \beta$.

    (b) Choose your own matrices α, $\beta \in M(\mathbb{Z}_{10})$ and compute $\det \alpha$, $\det \beta$, and $\det(\alpha \cdot \beta)$.

    (c) **Prove:** Let α, $\beta \in M(\mathbb{Z}_{10})$. Then $\det(\alpha \cdot \beta) = \det \alpha \cdot \det \beta$.

17. Let $\alpha \in M(\mathbb{Z}_{10})$ whose second row is a constant multiple of the first row. Explain why α does not have a multiplicative inverse.

← For an example, see Problem #13.