Abstract Algebra
Day 7 Class Work

1. Let α, $\beta \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix}5&6\\7&8\end{smallmatrix}\right]$.
   (a) We've seen that $\alpha + \beta = \left[\begin{smallmatrix}6&8\\0&2\end{smallmatrix}\right]$. This time, compute $\beta + \alpha$.
   (b) We've seen that $\alpha \cdot \beta = \left[\begin{smallmatrix}9&2\\3&0\end{smallmatrix}\right]$. This time, compute $\beta \cdot \alpha$. **Ans:** $\beta \cdot \alpha = \left[\begin{smallmatrix}3&4\\1&6\end{smallmatrix}\right]$.
   (c) Is addition in $M(\mathbb{Z}_{10})$ commutative? Why or why not?
   (d) Is multiplication in $M(\mathbb{Z}_{10})$ commutative? Why or why not?

2. Again, let α, $\beta \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix}5&6\\7&8\end{smallmatrix}\right]$.
   (a) Compute each of these sums: $\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right] + \alpha$, $\alpha + \left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right] + \beta$, $\beta + \left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$.
   (b) Explain why $\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$ is called the *additive identity* element of $M(\mathbb{Z}_{10})$. ← i.e, $\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right] \in M(\mathbb{Z}_{10})$ behaves like $0 \in \mathbb{Z}$.

3. Again, let α, $\beta \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix}5&6\\7&8\end{smallmatrix}\right]$.
   (a) Find $\tau \in M(\mathbb{Z}_{10})$ such that $\alpha + \tau = \left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$.
   (b) Find $\tau \in M(\mathbb{Z}_{10})$ such that $\tau + \beta = \left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$.
   (c) Given $\sigma \in M(\mathbb{Z}_{10})$, describe how you'd find $\tau \in M(\mathbb{Z}_{10})$ such that

$$\sigma + \tau = \left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right] \text{ and } \tau + \sigma = \left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right].$$

4. Verify that $M(\mathbb{Z}_{10})$ is a group under addition. ← Assume that addition in $M(\mathbb{Z}_{10})$ is associative.

5. Let $\varepsilon \in M(\mathbb{Z}_{10})$ where $\varepsilon = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$.
   (a) With $\alpha = \left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]$, compute the products $\varepsilon \cdot \alpha$ and $\alpha \cdot \varepsilon$.
   (b) **Optional:** With $\beta = \left[\begin{smallmatrix}5&6\\7&8\end{smallmatrix}\right]$, compute the products $\varepsilon \cdot \beta$ and $\beta \cdot \varepsilon$.
   (c) Explain why ε is called the *multiplicative identity* element of $M(\mathbb{Z}_{10})$. ← $\varepsilon \in M(\mathbb{Z}_{10})$ behaves like which element in $\mathbb{Z}$?

6. Let σ, $\tau \in M(\mathbb{Z}_{10})$ where $\sigma = \left[\begin{smallmatrix}2&1\\7&4\end{smallmatrix}\right]$ and $\tau = \left[\begin{smallmatrix}4&9\\3&2\end{smallmatrix}\right]$.
   (a) Compute the products $\sigma \cdot \tau$ and $\tau \cdot \sigma$.
   (b) How are the elements σ and τ related?

7. Determine if it's possible to find a matrix $\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]$ in $M(\mathbb{Z}_{10})$ such that $\left[\begin{smallmatrix}2&3\\0&0\end{smallmatrix}\right] \cdot \left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right] = \varepsilon$. If so, then find it. If not, then explain why not. ← Here, $\varepsilon = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$.

8. Is $M(\mathbb{Z}_{10})$ is a group under multiplication? Why or why not? **Ans:** It's not. (Why not?)

9. As in Problem #6, let σ, $\tau \in M(\mathbb{Z}_{10})$ where $\sigma = \left[\begin{smallmatrix}2&1\\7&4\end{smallmatrix}\right]$ and $\tau = \left[\begin{smallmatrix}4&9\\3&2\end{smallmatrix}\right]$. We saw that $\sigma \cdot \tau = \varepsilon$ and $\tau \cdot \sigma = \varepsilon$, so that σ and τ are *multiplicative inverses* of each other in $M(\mathbb{Z}_{10})$. Our friends are having the following conversation:

   Elizabeth: If we were given just $\sigma = \left[\begin{smallmatrix}2&1\\7&4\end{smallmatrix}\right]$, how could we have found τ on our own?

   Anita: Well, the diagonal entries of τ are the same as σ, but swapped. The 2 and 4 in σ become 4 and 2 and τ. But how do we get the 9 and 3 in τ?

   Elizabeth: Hmm, we have 1 and 7 in σ... I got it! *Negate* them to get the 9 and 3.

   Now suppose $\alpha \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix}6&7\\1&3\end{smallmatrix}\right]$. Find its multiplicative inverse β and verify your answer by computing $\alpha \cdot \beta$ and $\beta \cdot \alpha$. **Ans:** $\beta = \left[\begin{smallmatrix}3&3\\9&6\end{smallmatrix}\right]$.

10. Let $\alpha \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} 2 & 1 \\ 5 & 4 \end{smallmatrix}\right]$.
    (a) Explain why our friends might say that $\beta = \left[\begin{smallmatrix} 4 & 9 \\ 5 & 2 \end{smallmatrix}\right]$ is the multiplicative inverse of α.
    (b) Compute $\alpha \cdot \beta$. Were Elizabeth and Anita correct? **Ans:** Unfortunately, no.
    (c) Instead, let $\tau = 7 \cdot \beta$, i.e., multiply each entry of β by 7 (and reduce modulo 10). Now compute $\alpha \cdot \tau$ and $\tau \cdot \alpha$.

11. Let $\alpha \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} 2 & 3 \\ 1 & 5 \end{smallmatrix}\right]$. Find the multiplicative inverse of α.

    Note: Be sure to verify your answer!

12. Let $\alpha \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.
    (a) Describe how you'd find the multiplicative inverse of α.
    (b) Elizabeth asks, "We've seen matrices that don't have multiplicative inverses, like in Problem #7. How can we tell whether or not a matrix has a multiplicative inverse?"

13. Explain why it's impossible to find a matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in $M(\mathbb{Z}_{10})$ such that $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \varepsilon$. ← Here, $\varepsilon = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

14. Repeat Problem #13 with the matrix $\left[\begin{smallmatrix} 2 & 1 \\ 5 & 5 \end{smallmatrix}\right]$.

Definition: Let $\alpha \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. The *determinant* of α, denoted $\det\alpha$, is given by $\det\alpha = ad - bc$. Note that $\det\alpha$ is a number in $\mathbb{Z}_{10}$.

15. (a) Let $\alpha, \beta \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$. Find $\det\alpha$ and $\det\beta$. **Ans:** $\det\alpha = 8$.
    (b) Find the determinant of each matrix in Problems #9, #10, and #11.
    (c) While you're at it, find the determinants of the matrices in Problems #13 and #14.
    (d) What does the determinant of a matrix have to do with its multiplicative inverse?

16. Let $\alpha, \beta \in M(\mathbb{Z}_{10})$ where $\alpha = \left[\begin{smallmatrix} 2 & 1 \\ 5 & 4 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 3 & 4 \\ 1 & 2 \end{smallmatrix}\right]$.
    (a) Compute $\det\alpha$, $\det\beta$, and $\det(\alpha \cdot \beta)$. How are these determinants related?

        Note: For $\det(\alpha \cdot \beta)$, you must first find the product $\alpha \cdot \beta$.
    (b) Choose your own matrices $\alpha, \beta \in M(\mathbb{Z}_{10})$ and compute $\det\alpha$, $\det\beta$, and $\det(\alpha \cdot \beta)$.
    (c) **Prove:** Let $\alpha, \beta \in M(\mathbb{Z}_{10})$. Then $\det(\alpha \cdot \beta) = \det\alpha \cdot \det\beta$.

17. Let $\alpha \in M(\mathbb{Z}_{10})$ whose second row is a constant multiple of the first row. Explain why α does not have a multiplicative inverse. ← For an example, see Problem #13.