Abstract Algebra
Day 6 Class Work Solutions

1. Let σ, $\tau \in S_3$ defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Earlier, we computed $\sigma \circ \tau$, whose result is shown below on the left.

$$\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \text{ and } \tau \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}.$$

(a) Compute $\tau \circ \sigma$ by completing the matrix above.

Solution.

$$\tau \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

(b) Are $\sigma \circ \tau$ and $\tau \circ \sigma$ equal?

← Is this surprising?

Solution. No, $\sigma \circ \tau$ and $\tau \circ \sigma$ are *not* equal.

2. Let α, $\beta \in S_5$ defined by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}.$$

(a) Find α^{-1}.

Hint: Since $\alpha(1) = 4$, we can say that $\alpha^{-1}(4) = ?$

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

Solution.

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}.$$

(b) Find $\chi \in S_5$ such that $\alpha \circ \chi = \beta$. Here, χ is the inside function.

$$\chi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

Solution. We have

$$\chi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}.$$

For example, $(\alpha \circ \chi)(1) = \alpha(\chi(1)) = \alpha(2) = 1$ and $\beta(1) = 1$, so that $\alpha \circ \chi$ and β agree on the input 1. You should verify that $\alpha \circ \chi$ and β agree on inputs 2, 3, 4, and 5.

(c) Elizabeth claims that $\alpha^{-1} \circ \beta$ is equal to χ. Anita thinks $\chi = \beta \circ \alpha^{-1}$. Who do you think is correct? Explain your reasoning.

Solution. Elizabeth is correct. By composing both sides of $\alpha \circ \chi = \beta$ (on the left) by α^{-1}, we obtain $\chi = \alpha^{-1} \circ \beta$.

(d) Compute both $\alpha^{-1} \circ \beta$ and $\beta \circ \alpha^{-1}$ to see who was correct in part (d).

Solution. Computing $\alpha^{-1} \circ \beta$ yields the same permutation as χ that we found in part (b). We also have

$$\beta \circ \alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix},$$

which does *not* equal χ.

3. How many permutations are there in S_3? in S_4? in S_5? in S_n? Explain how you know.

Ans: 6 permutations in S_3.

Solution. Let $\sigma \in S_3$. There are 3 choices for $\sigma(1)$. For $\sigma(2)$, there are 2 choices, as $\sigma(2)$ cannot equal $\sigma(1)$. For $\sigma(3)$, there is only 1 choice, since $\sigma(3)$ cannot equal $\sigma(1)$ or $\sigma(2)$. Thus, there are $3! = 3 \cdot 2 \cdot 1 = 6$ permutations in S_3. Similarly, there are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ permutations in S_4, $5! = 120$ permutations in S_5, and $n!$ permutations in S_n.

4. Consider the following elements in S_4:

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

(a) Verify that $\gamma \circ \gamma = \varepsilon$. We say that the *order* of γ is 2, since $n = 2$ is the smallest positive exponent such that

$$\gamma^n = \underbrace{\gamma \circ \gamma \circ \cdots \circ \gamma}_{n \text{ copies}} = \varepsilon.$$

(b) Find the order of σ.

Ans for (b): $n = 2$.

Solution. The order of σ is 2.

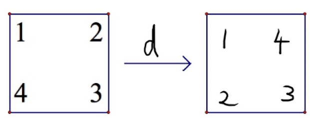
(c) Do the same for τ.

Solution. The order of τ is 3.

5. Elizabeth says, "S_4 and D_4 are the same, because when you shuffle 1, 2, 3, 4, that's like moving around the vertices of a square." Do you agree or disagree with her? Explain.

Hint: How many elements are in D_4? in S_4?

Solution. Disagree. There are 24 permutations in S_4 and only 8 symmetries in D_4, so they cannot be the same. We can view each symmetry (motion of a square) as shuffling the vertices for the square. For example, consider the reflection $d \in D_4$:



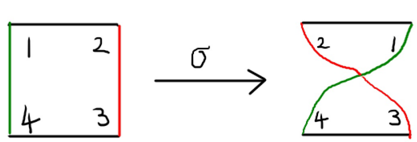
This reflection swaps the vertices 2 and 4, while keeping vertices 1 and 3 constant. Thus, we can view it as the analogous permutation

$$d = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$

But consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix},$$

which swaps 1 and 2, while keeping 3 and 4 constant. If we try to apply this to the same square, we end up twisting it, resulting in a shape that's not a square. (See figure below.) Thus, σ cannot be viewed as a valid symmetry.



6. Let $\sigma, \gamma \in S_3$ defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

(a) Construct a composition table for the set $H = \{\varepsilon, \sigma, \gamma\}$.

Solution.

$\circ$	ε	σ	γ
ε	ε	σ	γ
σ	σ	γ	ε
γ	γ	ε	σ

(b) Use the table created to check the group properties for H.

Solution.

1. H is closed under composition. Every entry in the table (i.e., all possible "products") is an element of H.
2. The associative law holds in S_3, and more generally in S_n. We'll soon see why.
3. H contains the identity element ε.
4. Every element in H has an inverse that's also in H. In fact, ε is a self inverse, while σ and γ are an inverse pair.

(c) Is H commutative or non-commutative?

Solution. H is commutative.

7. (a) If $\sigma \circ \tau = \sigma \circ \mu$ in S_n, show that $\tau = \mu$.

PROOF. Assume $\sigma \circ \tau = \sigma \circ \mu$ in S_n. Compose both sides of the equation *on the left* by σ^{-1} to obtain $\sigma^{-1} \circ (\sigma \circ \tau) = \sigma^{-1} \circ (\sigma \circ \mu)$. Using the associative law gives $(\sigma^{-1} \circ \sigma) \circ \tau = (\sigma^{-1} \circ \sigma) \circ \mu$. Since $\sigma^{-1} \circ \sigma = \varepsilon$, we get $\varepsilon \circ \tau = \varepsilon \circ \mu$. Finally, ε keeps all elements in S_n unchanged, i.e., $\varepsilon \circ \tau = \tau$ and $\varepsilon \circ \mu = \mu$. Thus, $\tau = \mu$ as desired. ■

(b) Does $\sigma \circ \tau = \mu \circ \sigma$ imply that $\tau = \mu$? Support your answer. **Ans for (b):** No. (Why?)

Solution. No. The issue here is that σ is on two different sides—it's on the left in $\sigma \circ \tau$, but it's on the right in $\mu \circ \sigma$. So, if we tried to compose both sides of the equation *on the left* by σ^{-1}, we get (after a few steps): $\tau = \sigma^{-1} \circ \mu \circ \sigma$. As a counterexample, consider these elements from S_3.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \mu = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

You should verify that $\sigma \circ \tau = \mu \circ \sigma$ is true. However, $\tau = \mu$ is false.

8. Find all subgroups of S_3. How do you know that you've found them all? ← There are six of them.

9. **(Some Food for Thought)**

(a) Can an element of S_5 have order greater than 5? If so, come up with such an element. If not, explain why not.

Recall: The *order* is the smallest positive exponent n described in Problem #4.

(b) What is the largest order that an element of S_5 can have? An element of S_6? S_7? S_n?

10. **(More Food for Thought)** Find all subgroups of S_4. How do you know that you've found them all? Can you generalize to S_n?