Abstract Algebra
Day 6 Class Work

1. Let σ, $\tau \in S_3$ defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Earlier, we computed $\sigma \circ \tau$, whose result is shown below on the left.

$$\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \text{ and } \tau \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}.$$

   (a) Compute $\tau \circ \sigma$ by completing the matrix above.

   (b) Are $\sigma \circ \tau$ and $\tau \circ \sigma$ equal? ← Is this surprising?

2. Let α, $\beta \in S_5$ defined by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}.$$

   (a) Find α^{-1}.

Hint: Since $\alpha(1) = 4$, we can say that $\alpha^{-1}(4) = ?$

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

   (b) Find $\chi \in S_5$ such that $\alpha \circ \chi = \beta$. Here, χ is the inside function.

$$\chi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

   (c) Elizabeth claims that $\alpha^{-1} \circ \beta$ is equal to χ. Anita thinks $\chi = \beta \circ \alpha^{-1}$. Who do you think is correct? Explain your reasoning.

   (d) Compute both $\alpha^{-1} \circ \beta$ and $\beta \circ \alpha^{-1}$ to see who was correct in part (d).

3. How many permutations are there in S_3? in S_4? in S_5? in S_n? Explain how you know.

Ans: 6 permutations in S_3.

4. Consider the following elements in S_4:

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \text{ and } \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

   (a) Verify that $\gamma \circ \gamma = \varepsilon$. We say that the *order* of γ is 2, since $n = 2$ is the smallest positive exponent such that

$$\gamma^n = \underbrace{\gamma \circ \gamma \circ \cdots \circ \gamma}_{n \text{ copies}} = \varepsilon.$$

   (b) Find the order of σ.

Ans for (b): $n = 2$.

   (c) Do the same for τ.

5. Elizabeth says, "S_4 and D_4 are the same, because when you shuffle 1, 2, 3, 4, that's like moving around the vertices of a square." Do you agree or disagree with her? Explain.

Hint: How many elements are in D_4? in S_4?

6. Let $\sigma,\, \gamma \in S_3$ defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

   (a) Construct a composition table for the set $H = \{\varepsilon,\, \sigma,\, \gamma\}$.

$\circ$	ε	σ	γ
ε			
σ			
γ			

   (b) Use the table created to check the group properties for H.

   (c) Is H commutative or non-commutative?

7. (a) If $\sigma \circ \tau = \sigma \circ \mu$ in S_n, show that $\tau = \mu$.

   (b) Does $\sigma \circ \tau = \mu \circ \sigma$ imply that $\tau = \mu$? Support your answer. **Ans for (b):** No. (Why?)

8. Find all subgroups of S_3. How do you know that you've found them all? ← There are six of them.

9. **(Some Food for Thought)**

   (a) Can an element of S_5 have order greater than 5? If so, come up with such an element. If not, explain why not.

   Recall: The *order* is the smallest positive exponent n described in Problem #4.

   (b) What is the largest order that an element of S_5 can have? An element of S_6? S_7? S_n?

10. **(More Food for Thought)** Find all subgroups of S_4. How do you know that you've found them all? Can you generalize to S_n?