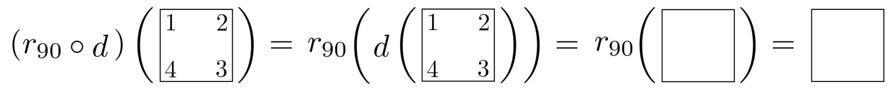
Abstract Algebra
Day 5 Class Work Solutions

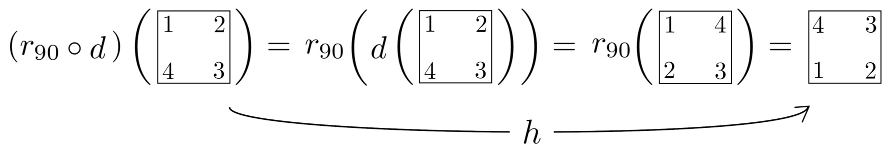
1. Consider the symmetries of a square $D_4 = \{\varepsilon, r_{90}, r_{180}, r_{270}, h, v, d, d'\}$.

   (a) Verify that $r_{90} \circ d = h$. In particular, note that $r_{90} \circ d \neq d \circ r_{90}$.

   ← Feel free to use the picture below.



   Solution. See figure below. Note that with $r_{90} \circ d$, the symmetry d is the inside function that gets applied to the square first.



   (b) Complete the composition table for D_4 below. For $\sigma, \tau \in D_4$, the "product" $\sigma \circ \tau$ is the entry in row σ and column τ.

   ← To check, see the entries for $r_{90} \circ d$ and $d \circ r_{90}$.

   Solution.

$\circ$	ε	r_{90}	r_{180}	r_{270}	h	v	d	d'
ε	ε	r_{90}	r_{180}	r_{270}	h	v	d	d'
r_{90}	r_{90}	r_{180}	r_{270}	ε	d'	d	h	v
r_{180}	r_{180}	r_{270}	ε	r_{90}	v	h	d'	d
r_{270}	r_{270}	ε	r_{90}	r_{180}	d	d'	v	h
h	h	d	v	d'	ε	r_{180}	r_{90}	r_{270}
v	v	d'	h	d	r_{180}	ε	r_{270}	r_{90}
d	d	v	d'	h	r_{270}	r_{90}	ε	r_{180}
d'	d'	h	d	v	r_{90}	r_{270}	r_{180}	ε

2. Now, we'll check the "group properties" of D_4. **Use the table from Problem #1(b) in answering the questions below.**

   ← Soon, we'll define what we mean by a "group."

   (a) Is D_4 closed under composition? Why or why not?

   Solution. Yes. We can see this from the table above, since every entry in the table (i.e., all possible "products") is an element of D_4.

   (b) Choose three elements $\sigma, \tau, \mu \in D_4$. **Using the table from Problem #1(b),** verify that $(\sigma \circ \tau) \circ \mu = \sigma \circ (\tau \circ \mu)$.

   ← i.e., the associative law.

   Solution. A possible example is:

$$(r_{90} \circ d) \circ r_{270} = h \circ r_{270} = d' \text{ and } r_{90} \circ (d \circ r_{270}) = r_{90} \circ h = d',$$

   so that $(r_{90} \circ d) \circ r_{270} = r_{90} \circ (d \circ r_{270})$.

   (c) When completing the composition table for D_4, why was it easy to compute the row and column containing ε? Why might ε be called the *identity* motion?

   Solution. We have $\varepsilon \circ \sigma = \sigma$ (first row of the table) and $\sigma \circ \varepsilon = \sigma$ (first column) for all $\sigma \in D_4$. Composing with ε is analogous to multiplying a number by 1.

(d) Look at the table again. Explain why r_{90} and r_{270} are said to be *inverses* of each other in D_4. Does every element in D_4 have an inverse? Why or why not?

Hint: $r_{90} \circ r_{270} = ??$

Solution. We have $r_{90} \circ r_{270} = \varepsilon$, so that r_{90} and r_{270} are an *inverse pair*. Yes, every element in D_4 have an inverse. The other inverse pairs are: $\varepsilon \circ \varepsilon = \varepsilon$, $r_{180} \circ r_{180} = \varepsilon$, $h \circ h = \varepsilon$, $v \circ v = \varepsilon$, $d \circ d = \varepsilon$, $d' \circ d' = \varepsilon$ (i.e., the rest are all self-inverses).

3. Consider $h \in D_4$, i.e., the reflection about the horizontal axis of the square.

(a) Let $C(h)$ be the subset of D_4 defined by $C(h) = \{\sigma \in D_4 \mid \sigma \circ h = h \circ \sigma\}$. For example, we have $\varepsilon \in C(h)$ because $\varepsilon \circ h = h \circ \varepsilon$. Find all elements of $C(h)$.

← $C(h)$ is called the *centralizer* of h in D_4.

(b) Check that $C(h) = \{\varepsilon, r_{180}, h, v\}$.

(c) Construct a composition table for $C(h)$, then check the "group properties" for $C(h)$. Why might $C(h)$ be called a sub*group* of D_4?

← Use the big table from Problem #1(b).

Solution.

$\circ$	ε	r_{180}	h	v
ε	ε	r_{180}	h	v
r_{180}	r_{180}	ε	v	h
h	h	v	ε	r_{180}
v	v	h	r_{180}	ε

- $C(h)$ is closed. We can see this from the table above, since every entry in the table (i.e., all possible "products") is an element of $C(h)$.
- The associative law holds in D_4, so it also holds for elements of $C(h)$.
- $C(h)$ contains the identity ε.
- Every element in $C(h)$ has an inverse in $C(h)$. In fact, each is a self-inverse.

4. (a) Come up with another subgroup of D_4. Be sure to check its "group properties."

(b) In fact, see if you can come up with all subgroups of D_4.

5. Pick any row or column of the table for D_4. Notice anything? Can you explain it?

← It's the Sudoku property.

Solution. Every row/column of the table contains every element of D_4 exactly once.

6. **(Some Food for Thought)**

(a) Draw a figure whose symmetries include only r_0, r_{90}, r_{180}, and r_{270}.

(b) Draw a figure whose symmetries include only r_0, r_{180}, h, and v.

(c) How many symmetries of a tetrahedron are there?

Note: *Tetrahedron* is a 3D figure with four equilateral triangles as its faces.