Abstract Algebra
Day 5 Class Work

1. Consider the symmetries of a square $D_4 = \{\varepsilon,\, r_{90},\, r_{180},\, r_{270},\, h,\, v,\, d,\, d'\}$.

   (a) Verify that $r_{90} \circ d = h$. In particular, note that $r_{90} \circ d \neq d \circ r_{90}$.

   ← Feel free to use the picture below.

$$(r_{90} \circ d)\left(\boxed{\begin{smallmatrix}1 & 2\\ 4 & 3\end{smallmatrix}}\right) = r_{90}\left(d\left(\boxed{\begin{smallmatrix}1 & 2\\ 4 & 3\end{smallmatrix}}\right)\right) = r_{90}\left(\boxed{\phantom{\begin{smallmatrix}1 & 2\\ 4 & 3\end{smallmatrix}}}\right) = \boxed{\phantom{\begin{smallmatrix}1 & 2\\ 4 & 3\end{smallmatrix}}}$$

   (b) Complete the composition table for D_4 below. For $\sigma,\, \tau \in D_4$, the "product" $\sigma \circ \tau$ is the entry in row σ and column τ.

   ← To check, see the entries for $r_{90} \circ d$ and $d \circ r_{90}$.

$\circ$	ε	r_{90}	r_{180}	r_{270}	h	v	d	d'
ε								
r_{90}			r_{270}		d'	d	h	v
r_{180}					v		d'	d
r_{270}					d	d'	v	h
h		d	v	d'		r_{180}	r_{90}	r_{270}
v		d'		d	r_{180}		r_{270}	r_{90}
d		v	d'	h	r_{270}	r_{90}		r_{180}
d'		h	d	v		r_{270}	r_{180}	

2. Now, we'll check the "group properties" of D_4. **Use the table from Problem #1(b) in answering the questions below.**

   ← Soon, we'll define what we mean by a "group."

   (a) Is D_4 closed under composition? Why or why not?

   (b) Choose three elements $\sigma,\, \tau,\, \mu \in D_4$. **Using the table from Problem #1(b),** verify that $(\sigma \circ \tau) \circ \mu = \sigma \circ (\tau \circ \mu)$.

   ← i.e., the associative law.

   (c) When completing the composition table for D_4, why was it easy to compute the row and column containing ε? Why might ε be called the *identity* motion?

   (d) Look at the table again. Explain why r_{90} and r_{270} are said to be *inverses* of each other in D_4. Does every element in D_4 have an inverse? Why or why not?

   Hint: $r_{90} \circ r_{270} = ??$

3. Consider $h \in D_4$, i.e., the reflection about the horizontal axis of the square.

   (a) Let $C(h)$ be the subset of D_4 defined by $C(h) = \{\sigma \in D_4 \mid \sigma \circ h = h \circ \sigma\}$. For example, we have $\varepsilon \in C(h)$ because $\varepsilon \circ h = h \circ \varepsilon$. Find all elements of $C(h)$.

   ← $C(h)$ is called the *centralizer* of h in D_4.

   (b) Check that $C(h) = \{\varepsilon,\, r_{180},\, h,\, v\}$.

   (c) Construct a composition table for $C(h)$, then check the "group properties" for $C(h)$. Why might $C(h)$ be called a sub*group* of D_4?

   ← Use the big table from Problem #1(b).

4. (a) Come up with another subgroup of D_4. Be sure to check its "group properties."

   (b) In fact, see if you can come up with all subgroups of D_4.

5. Pick any row or column of the table for D_4. Notice anything? Can you explain it?

   ← It's the Sudoku property.

6. **(Some Food for Thought)**

   (a) Draw a figure whose symmetries include only r_0, r_{90}, r_{180}, and r_{270}.

   (b) Draw a figure whose symmetries include only r_0, r_{180}, h, and v.

   (c) How many symmetries of a tetrahedron are there?

   Note: *Tetrahedron* is a 3D figure with four equilateral triangles as its faces.