Abstract Algebra
Day 4 Class Work

1. Compute each of the following in $\mathbb{Z}_7$. Simplify your result as much as possible.

   (a) $6 + 4$ (b) $4 \cdot 2$ (c) $2 - 5$ (d) 3^4 (e) 3258 (f) -3258

   Ans to (f): $-3258 = 4$.

2. (a) Describe all integers n such that $n = 0$ in $\mathbb{Z}_7$.

   ← Include *negative* integers.

   (b) Describe all integers n such that $n = 2$ in $\mathbb{Z}_7$.

3. For each pair a and b, determine whether or not $a = b$ in $\mathbb{Z}_7$.

   ← These should be done *without* a calculator.

   (a) $a = 16$ and $b = 30$

   (b) $a = 3258$ and $b = 3288$

   (c) $a = -710$ and $b = -731$

   (d) $a = 98765123406$ and $b = 98765123476$

4. Given a, $b \in \mathbb{Z}$, describe how you can determine whether or not $a = b$ in $\mathbb{Z}_7$. Can you do this *without* first simplifying each of a and b in $\mathbb{Z}_7$?

5. Consider $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ again. Recall that 3 is a multiplicative inverse of 5 because $3 \cdot 5 = 1$. (And vice versa, i.e., 5 is a multiplicative inverse of 3.) Find all other elements of $\mathbb{Z}_7$ that have multiplicative inverses.

   Ans: All of them except 0.

6. Now switch gears and consider the number system $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. Which one of these elements have multiplicative inverses?

   Ans: All of them except 0.

7. (a) Repeat Problem #6 with $\mathbb{Z}_6$, with $\mathbb{Z}_{10}$, and with $\mathbb{Z}_{15}$. Any conjectures?

   (b) In $\mathbb{Z}_{35}$, does 8 have a multiplicative inverse? What about 10? How do you know?

   ← You're *not* being asked to find these inverses.

8. (a) In $\mathbb{Z}_{2584}$, does 2583 have a multiple inverse? If so, find it. If not, explain why not.

   (b) Generalize your result from part (a).

9. (a) **True or False:** In $\mathbb{Z}_m$, if $a \cdot b = 0$, then $a = 0$ or $b = 0$.

   (b) The statement in part (a) is true for which values of m? false for which values of m?

   (c) *Justify* your conjectures from part (b).

10. In $\mathbb{Z}_7$, compute 6^{231}. Also compute 2^{101}.

    Ans: $2^{101} = 4$.

11. **(Some Food for Thought)** Compute

$$a^6 + a^5 + a^4 + a^3 + a^2 + a + 1$$

    for each $a \in \mathbb{Z}_7$ with $a \neq 1$. Can you explain what's going on and why?