Abstract Algebra
Day 37 Class Work Solutions

Let M be an ideal of a ring R. In the problems below, we'll prove the following claim:

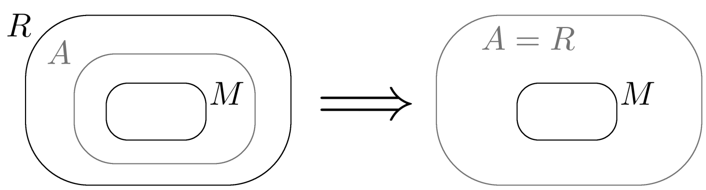
> **Claim.** If R/M is a field, then M is maximal in R.

1. (a) **(Discuss in your group)** Make sense of the following proof outline.
   - Let A be an ideal of R such that $M \subseteq A \subseteq R$.
   - We must show that $A = M$ or $A = R$.
   - If $A = M$, then we're done. So assume $A \neq M$.
   - Now we must show that $A = R$.

   Solution. Showing $A = M$ or $A = R$ would imply that there *isn't* an ideal strictly between M and R, and hence M is maximal.

   (b) Draw a picture that shows how M, A, and R from part (a) are related.

   Solution. Assuming $A \neq M$, we must show that $A = R$.



2. Let $a \in A$ such that $a \notin M$.

   (a) Based on the proof outline in Problem #1, explain why such an element a must exist.

   Solution. Such an element a must exist, because $M \subseteq A$ but $M \neq A$. Do you see where the element a would exist in the picture for Problem #1(b)?

   (b) In the quotient ring R/M, does $a + M = 0 + M$? Explain. **Ans:** No. (Why not?)

   Solution. No, $a \notin M$ implies that $a + M \neq 0 + M$.

3. (a) Explain why there exists $b + M \in R/M$ such that $(a + M) \cdot (b + M) = 1 + M$.

   (b) Before we could make the conclusion in part (a), why did we need to first verify (in Problem #2) that $a + M \neq 0 + M$? ← In a field, which elements have mult. inverses?

   Solution. Since R/M is a field, every *non-zero* element such as $a + M$ is a unit, i.e., it has a multiplicative inverse $b + M$ such that $(a + M) \cdot (b + M) = 1 + M$.

4. (a) Elizabeth says, "So $(a + M) \cdot (b + M) = 1 + M$ is the same as $a \cdot b + M = 1 + M$. That means $a \cdot b = 1$." Do you agree or disagree with her? Explain. ← Using the shortcut.

   Hint: $\alpha + M = \beta + M$ if and only if...?

   Solution. Disagree, since $a \cdot b + M = 1 + M$ implies that $a \cdot b - 1 \in M$.

   (b) How are the elements $a \cdot b$ and 1 *really* related?

   Solution. See solution to part (a).

(c) Let $m = a \cdot b - 1$. Explain why $m \in A$. **Ans to (c):** $m \in M \subseteq A$.

Solution. Since $m = a \cdot b - 1 \in M$ from above and $M \subseteq A$, we obtain $m \in A$.

(d) Use part (c) and the fact that A is an ideal to explain why $1 \in A$. **Hint:** $1 = a \cdot b - m$.

Solution. $1 = a \cdot b - m$ is in A, because a, $m \in A$ and A is an ideal, i.e., $a \cdot b \in A$ by product absorption; and $a \cdot b - m \in A$, because A is an additive subgroup of R.

5. Explain why $1 \in A$ implies that $A = R$.

Hint: $A \subseteq R$ is immediate, since A is an ideal of R. So it suffices to show that $R \subseteq A$. **Ans:** Suppose $r \in R$. Then $r = 1 \cdot r \in A$.

Solution. Let $r \in R$. Then, since $1 \in A$, we have $r = 1 \cdot r \in A$ by product absorption. Thus, $R \subseteq A$. Combining that with $A \subseteq R$, we obtain $A = R$.

6. **(Discuss in your group)** Review the proof of the claim again, making sure everyone in your group understands the key points.

Let M be an ideal of a ring R. In the problems below, we'll prove the following claim:

Claim. If M is maximal in R, then R/M is a field.

7. **(Discuss in your group)** Make sense of the following proof outline.

- Let $a + M \neq 0 + M$ in R/M. Thus, $a \notin M$.
- We must show that $(a + M) \cdot (r + M) = 1 + M$ for some $r + M \in R/M$.

Solution. To show that R/M is a field, we must show that any non-zero element of R/M such as $a + M$ defined above has a multiplicative inverse.

8. We have $a \notin M$. Define the set $M + \langle a \rangle = \{m + \alpha \mid m \in M,\ \alpha \in \langle a \rangle\}$, which turns out to be an ideal of R. Here, $\langle a \rangle = \{a \cdot r \mid r \in R\}$ is the principal ideal generated by a. ← Here, $M + \langle a \rangle$ is the sum of two ideals.

(a) Explain why $M \subseteq M + \langle a \rangle$.

Hint: Let $m \in M$ and show that $m \in M + \langle a \rangle$. **Ans:** $m = m + a \cdot 0$.

Solution. Let $m \in M$. Then $m = m + a \cdot 0 \in M + \langle a \rangle$.

(b) Explain why $a \in M + \langle a \rangle$. **Ans:** $a = 0 + a \cdot 1$.

Solution. We have $a = 0 + a \cdot 1 \in M + \langle a \rangle$. Note that $0 \in M$, because M is an ideal, hence an additive subgroup of R.

9. Problem #8(a) shows that $M \subseteq M + \langle a \rangle \subseteq R$.

(a) Given that M is a maximal ideal, what can we conclude about $M + \langle a \rangle$?

Solution. Either $M + \langle a \rangle = M$ or $M + \langle a \rangle = R$.

(b) Does $M + \langle a \rangle = M$ or $M + \langle a \rangle = R$?

Hint: Your result in Problem #8(b) should help.

Solution. Since $a \in M + \langle a \rangle$, but $a \notin M$, we have $M + \langle a \rangle \neq M$. Thus, $M + \langle a \rangle = R$.

10. (a) Use Problem #9(b) to explain why $1 = m + a \cdot r$ for some $m \in M$ and $r \in R$.

    Solution. Since $1 \in R$, and $M + \langle a \rangle = R$, we have $1 \in M + \langle a \rangle$.

    (b) Explain why $(a + M) \cdot (r + M) = 1 + M$. ← And the proof is done!

    Solution. Since $1 = m + a \cdot r$, we have $1 - a \cdot r = m \in M$, so that $a \cdot r + M = 1 + M$. But $a \cdot r + M = (a + M) \cdot (r + M)$, so that $(a + M) \cdot (r + M) = 1 + M$.

11. **(Discuss in your group)** Review the proof of the claim again, making sure everyone in your group understands the key points.

12. In the above proof, we used the fact that

$$M + \langle a \rangle = \{m + \alpha \mid m \in M,\, \alpha \in \langle a \rangle\}$$

is an ideal of R. Note that M and $\langle a \rangle$ are both ideals of R.

Prove: Let I and J be ideals of a ring R. Define ← From Chapter 31.

$$I + J = \{i + j \mid i \in I,\, j \in J\}.$$

Then $I + J$ is an ideal of R.

Proof. We first show that $I + J$ is an additive subgroup of R. For closure, let $i_1 + j_1$, $i_2 + j_2 \in I + J$ where i_1, $i_2 \in I$ and j_1, $j_2 \in J$. Then,

$$(i_1 + j_1) + (i_2 + j_2) = (i_1 + i_2) + (j_1 + j_2) \in I + J,$$

since I and J are both closed under addition. Since I and J both contain the additive identity 0, we have $0 = 0 + 0 \in I + J$. Lastly, $-(i_1 + j_1) = (-i_1) + (-j_1) \in I + J$ so that $I + J$ contains the additive inverses of its elements.

For product absorption, let $r \in R$. Then $r \cdot (i_1 + j_1) = (r \cdot i_1) + (r \cdot j_1) \in I + J$, since I and J both satisfy product absorption. Therefore, $I + J$ is an ideal of R, as desired. ■

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13. Write down complete proofs of today's claims: ← M is an ideal of a ring R.

    (a) If R/M is a field, then M is maximal in R.

    (b) If M is maximal in R, then R/M is a field.

    Solution. See Theorems 37.2(a) and 37.2(b) and their proofs in the textbook.

14. In Problem #13(a), we proved the contrapositive of the statement:

    If M is *not* maximal in R, then R/M is *not* a field.

    Explore and describe what happens if we try to prove the original statement, rather than the contrapositive.

15. In Problem #13(a), we assumed that $A \neq M$ and showed that $A = R$. Explore and describe what happens if we assume instead that $A \neq R$ and try to show that $A = M$.

16. After writing down the proof in Problem #13(b), Anita wonders:

    "$M + \langle a \rangle$ seems like it was pulled out of thin air. What's the *motivation* behind using this ideal? How could I have come up with it on my own?"

    How would you respond to Anita?

    Solution. See the explanation after the proof of Theorem 37.2(b) in the textbook.