Abstract Algebra
Day 37 Class Work

Let M be an ideal of a ring R. In the problems below, we'll prove the following claim:

> **Claim.** If R/M is a field, then M is maximal in R.

1. (a) **(Discuss in your group)** Make sense of the following proof outline.
   - Let A be an ideal of R such that $M \subseteq A \subseteq R$.
   - We must show that $A = M$ or $A = R$.
   - If $A = M$, then we're done. So assume $A \neq M$.
   - Now we must show that $A = R$.

   (b) Draw a picture that shows how M, A, and R from part (a) are related.

2. Let $a \in A$ such that $a \notin M$.

   (a) Based on the proof outline in Problem #1, explain why such an element a must exist.

   (b) In the quotient ring R/M, does $a + M = 0 + M$? Explain. — **Ans:** No. (Why not?)

3. (a) Explain why there exists $b + M \in R/M$ such that $(a + M) \cdot (b + M) = 1 + M$.

   (b) Before we could make the conclusion in part (a), why did we need to first verify (in Problem #2) that $a + M \neq 0 + M$? — ← In a field, which elements have mult. inverses?

4. (a) Elizabeth says, "So $(a + M) \cdot (b + M) = 1 + M$ is the same as $a \cdot b + M = 1 + M$. That means $a \cdot b = 1$." Do you agree or disagree with her? Explain. — ← Using the shortcut.

   Hint: $\alpha + M = \beta + M$ if and only if...?

   (b) How are the elements $a \cdot b$ and 1 *really* related?

   (c) Let $m = a \cdot b - 1$. Explain why $m \in A$. — **Ans to (c):** $m \in M \subseteq A$.

   (d) Use part (c) and the fact that A is an ideal to explain why $1 \in A$. — **Hint:** $1 = a \cdot b - m$.

5. Explain why $1 \in A$ implies that $A = R$.

   Hint: $A \subseteq R$ is immediate, since A is an ideal of R. So it suffices to show that $R \subseteq A$. — **Ans:** Suppose $r \in R$. Then $r = 1 \cdot r \in A$.

6. **(Discuss in your group)** Review the proof of the claim again, making sure everyone in your group understands the key points.

Let M be an ideal of a ring R. In the problems below, we'll prove the following claim:

> **Claim.** If M is maximal in R, then R/M is a field.

7. **(Discuss in your group)** Make sense of the following proof outline.
   - Let $a + M \neq 0 + M$ in R/M. Thus, $a \notin M$.
   - We must show that $(a + M) \cdot (r + M) = 1 + M$ for some $r + M \in R/M$.

8. We have $a \notin M$. Define the set $M + \langle a \rangle = \{m + \alpha \mid m \in M, \, \alpha \in \langle a \rangle\}$, which turns out to be an ideal of R. Here, $\langle a \rangle = \{a \cdot r \mid r \in R\}$ is the principal ideal generated by a. ← Here, $M + \langle a \rangle$ is the sum of two ideals.

   (a) Explain why $M \subseteq M + \langle a \rangle$.

   Hint: Let $m \in M$ and show that $m \in M + \langle a \rangle$. **Ans:** $m = m + a \cdot 0$.

   (b) Explain why $a \in M + \langle a \rangle$. **Ans:** $a = 0 + a \cdot 1$.

9. Problem #8(a) shows that $M \subseteq M + \langle a \rangle \subseteq R$.

   (a) Given that M is a maximal ideal, what can we conclude about $M + \langle a \rangle$?

   (b) Does $M + \langle a \rangle = M$ or $M + \langle a \rangle = R$?

   Hint: Your result in Problem #8(b) should help.

10. (a) Use Problem #9(b) to explain why $1 = m + a \cdot r$ for some $m \in M$ and $r \in R$.

    (b) Explain why $(a + M) \cdot (r + M) = 1 + M$. ← And the proof is done!

11. **(Discuss in your group)** Review the proof of the claim again, making sure everyone in your group understands the key points.

12. In the above proof, we used the fact that

$$M + \langle a \rangle = \{m + \alpha \mid m \in M, \, \alpha \in \langle a \rangle\}$$

is an ideal of R. Note that M and $\langle a \rangle$ are both ideals of R.

Prove: Let I and J be ideals of a ring R. Define ← From Chapter 31.

$$I + J = \{i + j \mid i \in I, \, j \in J\}.$$

Then $I + J$ is an ideal of R.

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13. Write down complete proofs of today's claims: ← M is an ideal of a ring R.

    (a) If R/M is a field, then M is maximal in R.

    (b) If M is maximal in R, then R/M is a field.

14. In Problem #13(a), we proved the contrapositive of the statement:

    If M is *not* maximal in R, then R/M is *not* a field.

    Explore and describe what happens if we try to prove the original statement, rather than the contrapositive.

15. In Problem #13(a), we assumed that $A \neq M$ and showed that $A = R$. Explore and describe what happens if we assume instead that $A \neq R$ and try to show that $A = M$.

16. After writing down the proof in Problem #13(b), Anita wonders:

    "$M + \langle a \rangle$ seems like it was pulled out of thin air. What's the *motivation* behind using this ideal? How could I have come up with it on my own?"

    How would you respond to Anita?