Abstract Algebra
Day 36 Class Work

1. Determine whether each ideal of $\mathbb{Z}$ is maximal.

   (a) $6\mathbb{Z}$ (b) $17\mathbb{Z}$ (c) $41\mathbb{Z}$ (d) $375\mathbb{Z}$ (e) $n\mathbb{Z}$

2. Determine whether each ideal of $\mathbb{R}[x]$ is maximal. You don't have to justify these (yet). ← Your *instinct* says. . . ?

   (a) $\langle x^2-1\rangle = \{(x^2-1)\cdot q(x) \mid q(x) \in \mathbb{R}[x]\}$, i.e., the set of all multiples of x^2-1.

   (b) $\langle x^2+1\rangle$

3. Let $g(x) = x^2-1 \in \mathbb{R}[x]$. Below, we'll show that $\langle g(x)\rangle$ is *not* maximal in $\mathbb{R}[x]$.

   (a) **Prove:** $\langle g(x)\rangle \subseteq \langle x+1\rangle$. ← Just like $12\mathbb{Z} \subseteq 4\mathbb{Z}$.

   Hint: Let $\alpha(x) \in \langle g(x)\rangle$ and show that $\alpha(x) \in \langle x+1\rangle$.

   (b) Find a polynomial $\beta(x)$ such that $\beta(x) \in \langle x+1\rangle$, but $\beta(x) \notin \langle g(x)\rangle$. **Ans:** $\beta(x) = x+1$.

   (c) Find a polynomial $\gamma(x)$ such that $\gamma(x) \in \mathbb{R}[x]$, but $\gamma(x) \notin \langle x+1\rangle$.

   (d) Use your results above to explain why

   $$\langle g(x)\rangle \subsetneq \langle x+1\rangle \subsetneq \mathbb{R}[x],$$

   i.e., $\langle x+1\rangle$ is *strictly* between $\langle g(x)\rangle$ and $\mathbb{R}[x]$. **So, $\langle g(x)\rangle$ is *not* maximal in $\mathbb{R}[x]$.**

   Recall: $\langle g(x)\rangle \subsetneq \langle x+1\rangle$ means $\langle g(x)\rangle \subseteq \langle x+1\rangle$, but $\langle g(x)\rangle \neq \langle x+1\rangle$.

Recall. Every ideal of $F[x]$ is principal, i.e., $A = \langle p(x)\rangle$ for some $p(x) \in F[x]$.

← See Chapter 31.

4. Let $g(x) = x^2+1 \in \mathbb{R}[x]$. Below, we'll show that $\langle g(x)\rangle$ is maximal in $\mathbb{R}[x]$.

   (a) Elizabeth says, "Here's my plan. Let $\langle p(x)\rangle$ be an ideal where $\langle g(x)\rangle \subseteq \langle p(x)\rangle \subseteq \mathbb{R}[x]$. Then I'll show that $\langle p(x)\rangle$ must be equal to either $\langle g(x)\rangle$ or $\mathbb{R}[x]$." Explain why her approach will show that $\langle g(x)\rangle$ is maximal. **Ans:** This is the definition of a *maximal* ideal.

   (b) **Prove:** If $\langle g(x)\rangle \subseteq \langle p(x)\rangle$, then $g(x) = p(x)\cdot q(x)$ for some $q(x) \in \mathbb{R}[x]$. **Hint:** $g(x) \in \langle g(x)\rangle$.

   (c) Anita says, "But $g(x) = x^2+1$ is unfactorable in $\mathbb{R}[x]$. So $g(x) = p(x)\cdot q(x)$ would mean that either $p(x)$ or $q(x)$ has to be a constant." What might she mean?

   (d) Suppose $p(x)$ is a constant. Say $p(x) = 3$, so that $g(x) = p(x)\cdot q(x)$ would be $x^2+1 = 3\cdot\left(\frac{1}{3}x^2+\frac{1}{3}\right)$. Explain why $\langle 3\rangle = \mathbb{R}[x]$, so that $\langle p(x)\rangle = \mathbb{R}[x]$.

   Hint: $\langle 3\rangle = \mathbb{R}[x]$ means every polynomial in $\mathbb{R}[x]$ is a multiple of 3. **Ans:** $f(x) = 3\cdot\left(\frac{1}{3}f(x)\right)$.

   (e) This time, suppose $q(x)$ is a constant. Say $q(x) = 3$ and $p(x) = \frac{1}{3}x^2+\frac{1}{3}$, so that $g(x) = p(x)\cdot q(x)$ would be $x^2+1 = \left(\frac{1}{3}x^2+\frac{1}{3}\right)\cdot 3$.

   Explain why $\langle \frac{1}{3}x^2+\frac{1}{3}\rangle = \langle x^2+1\rangle$, so that $\langle p(x)\rangle = \langle g(x)\rangle$.

   Hint: Show that $\langle \frac{1}{3}x^2+\frac{1}{3}\rangle \subseteq \langle x^2+1\rangle$ and $\langle x^2+1\rangle \subseteq \langle \frac{1}{3}x^2+\frac{1}{3}\rangle$.

   (f) Explain why $\langle g(x)\rangle$ is a maximal ideal.

5. Generalize your results from Problems #3 and #4 to prove each of the following.

   Theorem. Let F be a field and fix $g(x) \in F[x]$.

   (a) If $g(x)$ is factorable, then $\langle g(x)\rangle$ is *not* maximal in $F[x]$.

   (b) If $g(x)$ is unfactorable, then $\langle g(x)\rangle$ is maximal in $F[x]$.

6. **Prove:** The ideal $n\mathbb{Z}$ is maximal in $\mathbb{Z}$ if and only if n is prime.

   Hint: This is an "if and only if" statement, so there are two directions to prove. For one of them, it would be easier to prove the contrapositive. ← Your polynomial proofs should help, too.

7. Consider the ring $\mathbb{Z}_{12}$.
   (a) Find all of its additive subgroups. ← There are six of them.
   (b) Verify that each subgroup in part (a) satisfies the product absorption property. ← Thus, they're ideals.
   (c) Determine which ideals of $\mathbb{Z}_{12}$ are maximal. **Ans:** $\langle 3 \rangle$ and $\langle 2 \rangle$.

8. Repeat Problem #7 with the ring $\mathbb{Z}_7$.

9. In Problem #7, we saw that $\mathbb{Z}_{12}$ has exactly two maximal ideals.
   (a) Verify that $\mathbb{Z}_{20}$ has exactly two maximal ideals.
   (b) Verify that $\mathbb{Z}_{28}$ has exactly two maximal ideals.
   (c) Verify that $\mathbb{Z}_{18}$ has exactly two maximal ideals.
   (d) Find a few more values of n for which $\mathbb{Z}_n$ has exactly two maximal ideals.
   (e) What conjectures do you have?

10. Find a ring that has exactly three maximal ideals.

11. (a) Verify that 9 is *not* a unit in $\mathbb{Z}_{24}$. Then find a maximal ideal of $\mathbb{Z}_{24}$ containing 9.
   (b) Verify that 10 is *not* a unit in $\mathbb{Z}_{35}$. Then find a maximal ideal of $\mathbb{Z}_{35}$ containing 10.
   (c) Find a non-unit in $\mathbb{Z}_{30}$ and a maximal ideal of $\mathbb{Z}_{30}$ containing that non-unit element.
   (d) Find a non-unit in $\mathbb{Z}_{54}$ and a maximal ideal of $\mathbb{Z}_{54}$ containing that non-unit element.
   (e) What conjecture do you have?