Abstract Algebra
Day 34 Class Work Solutions

For the problems below, fix $x^2+1 \in \mathbb{R}[x]$ and define $\langle x^2+1\rangle = \{(x^2+1)\cdot q(x) \mid q(x) \in \mathbb{R}[x]\}$.

← i.e., the principal ideal generated by x^2+1.

1. Let $f(x) = 5x^4 + x^3 - 3x^2 + 4x - 3 \in \mathbb{R}[x]$.

   (a) Given the Mathematica output below, describe how the polynomials $f(x)$, x^2+1, $-8+x+5x^2$, and $5+3x$ are related to each other.

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In[18]:= f = 5 x^4 + x^3 - 3 x^2 + 4 x - 3

In[19]:= PolynomialQuotientRemainder[f, x^2 + 1, x]

Out[19]= {-8 + x + 5 x^2, 5 + 3 x}
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   Solution. We have $f(x) = (x^2+1)\cdot(-8+x+5x^2) + (5+3x)$. With $d(x) = x^2+1$ and $r(x) = 5+3x$, note how $\deg r(x) < \deg d(x)$.

   (b) Find $g(x) \in \mathbb{R}[x]$ of the smallest degree such that $f(x) + \langle x^2+1\rangle = g(x) + \langle x^2+1\rangle$.

   Ans: $g(x) = 5+3x$.

   Solution. Let $g(x) = 5+3x$. Then $f(x) - g(x) = (x^2+1)\cdot(-8+x+5x^2) \in \langle x^2+1\rangle$. Thus, $f(x) + \langle x^2+1\rangle = g(x) + \langle x^2+1\rangle$.

2. (a) Let $f(x) \in \mathbb{R}[x]$. Explain why $f(x) + \langle x^2+1\rangle$ can be "reduced" to $(a+bx) + \langle x^2+1\rangle$ where a, $b \in \mathbb{R}$.

   ← What are the remainders when dividing by x^2+1?

   Solution. By the division algorithm, we have $f(x) = (x^2+1)\cdot q(x) + (a+bx)$ for some $q(x) \in \mathbb{R}[x]$. Note here that the remainder $a+bx$ has smaller degree than the divisor x^2+1. Now let $g(x) = a+bx$. Then $f(x) - g(x) = (x^2+1)\cdot q(x) \in \langle x^2+1\rangle$. Thus, $f(x) + \langle x^2+1\rangle = g(x) + \langle x^2+1\rangle$.

   (b) Describe all distinct elements of $\mathbb{R}[x]/\langle x^2+1\rangle$.

   Solution. We have $\mathbb{R}[x]/\langle x^2+1\rangle = \{(a+bx) + \langle x^2+1\rangle \mid a, b \in \mathbb{R}\}$.

3. (a) In $\mathbb{R}[x]/\langle x^2+1\rangle$, explain why $x^2 + \langle x^2+1\rangle = -1 + \langle x^2+1\rangle$.

   Hint: $f(x) + \langle x^2+1\rangle = g(x) + \langle x^2+1\rangle$ if and only if... ?

   Solution. We have $x^2 + \langle x^2+1\rangle = -1 + \langle x^2+1\rangle$, because

   $$x^2 - (-1) = x^2 + 1 \in \langle x^2+1\rangle.$$

   (b) Let $f(x) = 5x^4 + x^3 - 3x^2 + 4x - 3 \in \mathbb{R}[x]$ again. After completing part (a) above, Elizabeth wrote down the following calculation:

   ← We can treat x^2 and -1 to be the same *as coset representatives*.

   $$\begin{aligned} f(x) + \langle x^2+1\rangle &= (5x^4 + x^3 - 3x^2 + 4x - 3) + \langle x^2+1\rangle \\ &= (5\cdot \boldsymbol{x^2}\cdot \boldsymbol{x^2} + \boldsymbol{x^2}\cdot x - 3\cdot \boldsymbol{x^2} + 4x - 3) + \langle x^2+1\rangle \\ &= (5\cdot \mathbf{(-1)}\cdot \mathbf{(-1)} + \mathbf{(-1)}\cdot x - 3\cdot \mathbf{(-1)} + 4x - 3) + \langle x^2+1\rangle \end{aligned}$$

   Complete her calculation to verify that $f(x) + \langle x^2+1\rangle = (5+3x) + \langle x^2+1\rangle$.

   ← Compare with Prob. #1. Which do you prefer?

   Solution. We have $5\cdot(-1)\cdot(-1) + (-1)\cdot x - 3\cdot(-1) + 4x - 3 = 5+3x$. Thus, $f(x) + \langle x^2+1\rangle = (5+3x) + \langle x^2+1\rangle$.

4. Compute each product and reduce it using Elizabeth's method from Problem #3:

   (a) $((2+7x) + \langle x^2+1\rangle)\cdot((4+3x) + \langle x^2+1\rangle)$

Solution.

$$\begin{aligned}((2+7x)+\langle x^2+1\rangle)\cdot((4+3x)+\langle x^2+1\rangle) &= (2+7x)(4+3x)+\langle x^2+1\rangle \\ &= (8+34x+21\cdot \boldsymbol{x^2})+\langle x^2+1\rangle \\ &= (8+34x+21\cdot \mathbf{(-1)})+\langle x^2+1\rangle \\ &= (-13+34x)+\langle x^2+1\rangle\end{aligned}$$

(b) $((-1+2x)+\langle x^2+1\rangle)\cdot((3+5x)+\langle x^2+1\rangle)$

Solution.

$$\begin{aligned}((-1+2x)+\langle x^2+1\rangle)\cdot((3+5x)+\langle x^2+1\rangle) &= (-1+2x)(3+5x)+\langle x^2+1\rangle \\ &= (-3+x+10\cdot \boldsymbol{x^2})+\langle x^2+1\rangle \\ &= (-3+x+10\cdot \mathbf{(-1)})+\langle x^2+1\rangle \\ &= (-13+x)+\langle x^2+1\rangle\end{aligned}$$

5. Compute each product in $\mathbb{C}=\{a+bi \mid a,\, b\in\mathbb{R}\}$, the *field* of complex numbers.

   Note: Recall that $i\cdot i=-1$.

   (a) $(2+7i)\cdot(4+3i)$

   Solution. $(2+7i)\cdot(4+3i)=8+34i+21\cdot \boldsymbol{i^2}=8+34i+21\cdot\mathbf{(-1)}=-13+34i.$

   (b) $(-1+2i)\cdot(3+5i)$

   Solution. $(-1+2i)\cdot(3+5i)=-3+i+10\cdot \boldsymbol{i^2}=-3+i+10\cdot\mathbf{(-1)}=-13+i.$

6. (a) Compare the description of the distinct elements of $\mathbb{R}[x]/\langle x^2+1\rangle$ that you gave in Problem #2(b) with $\mathbb{C}=\{a+bi \mid a,\, b\in\mathbb{R}\}$.

   (b) Compare your answers in Problem #4 with those in Problem #5.

   (c) To which familiar ring is $\mathbb{R}[x]/\langle x^2+1\rangle$ isomorphic? Explain.

   (d) Is $\mathbb{R}[x]/\langle x^2+1\rangle$ a field? Why or why not? **Ans:** Yes, because $\mathbb{C}$ is.

   Solution. The rings $\mathbb{C}$ and $\mathbb{R}[x]/\langle x^2+1\rangle$ are isomorphic, with $a+bi\in\mathbb{C}$ corresponding to the coset $(a+bx)+\langle x^2+1\rangle$ in $\mathbb{R}[x]/\langle x^2+1\rangle$. And since $\mathbb{C}$ is a field, so is $\mathbb{R}[x]/\langle x^2+1\rangle$. See Theorem 34.4 in the textbook for more details about this isomorphism.

7. Since $\mathbb{R}[x]/\langle x^2+1\rangle$ is a field, every nonzero element is a unit. Find $((3+4x)+\langle x^2+1\rangle)^{-1}$, i.e., find $(a+bx)+\langle x^2+1\rangle$ such that

   Ans: $(\frac{3}{25}-\frac{4}{25}x)+\langle x^2+1\rangle$.

   $$((3+4x)+\langle x^2+1\rangle)\cdot((a+bx)+\langle x^2+1\rangle)=1+\langle x^2+1\rangle.$$

   Note: If you're stuck, look ahead at Problem #8.

   Solution. The coset $(3+4x)+\langle x^2+1\rangle$ corresponds to $3+4i\in\mathbb{C}$, whose multiplicative inverse is $(3+4i)^{-1}=\frac{3}{25}-\frac{4}{25}i$. (See Problem #8 below.) Translating back to $\mathbb{R}[x]/\langle x^2+1\rangle$, we obtain the multiplicative inverse $\left(\frac{3}{25}-\frac{4}{25}x\right)+\langle x^2+1\rangle$.

8. (a) Compute the product $(3+4i)\cdot(3-4i)$. **Ans to (a):** 25.

Solution. Note that $3-4i$ is the *complex conjugate* of $3+4i$. We have

$$\begin{aligned}(3+4i)\cdot(3-4i) &= 3\cdot 3+3\cdot(-4i)+4i\cdot 3+4i\cdot(-4i)\\ &= 3^2-4^2\cdot \boldsymbol{i^2} \qquad \leftarrow \text{ the terms } -12i \text{ and } 12i \text{ cancel each other}\\ &= 3^2-4^2\cdot\boldsymbol{(-1)}\\ &= 3^2+4^2,\end{aligned}$$

so that $(3+4i)\cdot(3-4i)=3^2+4^2=25$.

(b) Find real numbers a and b such that

$$\frac{1}{3+4i}\cdot\frac{3-4i}{3-4i}=a+bi.$$

Solution. Using $(3+4i)\cdot(3-4i)=25$ from part (a), we have

$$\frac{1}{3+4i}\cdot\frac{3-4i}{3-4i}=\frac{3-4i}{(3+4i)\cdot(3-4i)}=\frac{3-4i}{25}=\frac{3}{25}-\frac{4}{25}i.$$

Therefore, $a=\frac{3}{25}$ and $b=-\frac{4}{25}$.

(c) Find $(3+4i)^{-1}$ in $\mathbb{C}$.

Hint: You already found it.

Solution. Since $3+4i$ is a non-zero (complex) number, its reciprocal $\frac{1}{3+4i}$ is the multiplicative inverse of $3+4i$. From part (b), we have $(3+4i)^{-1}=\frac{3}{25}-\frac{4}{25}i$.

(d) Compare with Problem #7. What's going on here?

Solution. See solution to Problem #7.

9. Find $((5+2x)+\langle x^2+1\rangle)^{-1}$ using the technique from Problem #8.

Solution. The coset $(5+2x)+\langle x^2+1\rangle$ corresponds to $5+2i\in\mathbb{C}$, whose multiplicative inverse is

$$(5+2i)^{-1}=\frac{1}{5+2i}\cdot\frac{5-2i}{5-2i}=\frac{5-2i}{29}=\frac{5}{29}-\frac{2}{29}i.$$

This corresponds to $((5+2x)+\langle x^2+1\rangle)^{-1}=\left(\frac{5}{29}-\frac{2}{29}x\right)+\langle x^2+1\rangle$.

10. Define a function $\theta:\mathbb{C}\to\mathbb{R}[x]/\langle x^2+1\rangle$ where $\theta(a+bi)=(a+bx)+\langle x^2+1\rangle$ for all $a+bi\in\mathbb{C}$. Show that θ is a ring homomorphism.

Note: It turns out that θ is a bijection. Thus, θ is in fact an *isomorphism*.

← Do you see why?

11. Prove that $\mathbb{C}$ is a field by showing that $(a+bi)^{-1}$ exists for all $a+bi\neq 0$.

Solution. The calculations in Problems #8 and #9 can be generalized to show that every non-zero element of $\mathbb{C}$ has a multiplicative inverse. I'll leave the details to you!

12. Let F be a field and fix $g(x)\in F[x]$. Prove each of the following:

(a) If $g(x)$ is factorable, then $F[x]/\langle g(x)\rangle$ is *not* a field.

← See if you can reproduce the proof from Day 33.

(b) **(Optional challenge)** If $g(x)$ is unfactorable, then $F[x]/\langle g(x)\rangle$ is a field.

13. Let $f(x)=4x^5+5x^4+x^3+9x^2-3x+4\in\mathbb{R}[x]$.

(a) Verify that $f(i)=0$ where $i=\sqrt{-1}\in\mathbb{C}$.

(b) Verify that $f(-i)=0$ as well.

14. (a) Repeat Problem #13, this time with $f(x) = 7x^{12} + 7x^{10} - 3x^5 - 3x^3 + 2x^2 + 2$.

    (b) **Prove:** Let $f(x) \in \mathbb{R}[x]$. If $i \in \mathbb{C}$ is a root of $f(x)$, then $-i$ is also a root of $f(x)$.

15. Consider the function $\varphi : \mathbb{R}[x] \to \mathbb{C}$ where $\varphi(f(x)) = f(i)$ for all $f(x) \in \mathbb{R}[x]$. Prove that:

    (a) φ is a ring homomorphism.

    (b) $\ker \varphi = \langle x^2 + 1 \rangle$. (**Hint:** Use Problem #14b.)

    (c) $\operatorname{im} \varphi = \mathbb{C}$.

    (d) What conclusion can you make using the First Isomorphism Theorem?