Abstract Algebra
Day 34 Class Work

For the problems below, fix $x^2+1 \in \mathbb{R}[x]$ and define $\langle x^2+1\rangle = \{(x^2+1)\cdot q(x) \mid q(x) \in \mathbb{R}[x]\}$. ← i.e., the principal ideal generated by x^2+1.

1. Let $f(x) = 5x^4 + x^3 - 3x^2 + 4x - 3 \in \mathbb{R}[x]$.

   (a) Given the Mathematica output below, describe how the polynomials $f(x)$, x^2+1, $-8+x+5x^2$, and $5+3x$ are related to each other.

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In[18]:= f = 5 x^4 + x^3 - 3 x^2 + 4 x - 3

In[19]:= PolynomialQuotientRemainder[f, x^2 + 1, x]

Out[19]= {-8 + x + 5 x^2, 5 + 3 x}
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   (b) Find $g(x) \in \mathbb{R}[x]$ of the smallest degree such that $f(x) + \langle x^2+1\rangle = g(x) + \langle x^2+1\rangle$. **Ans:** $g(x) = 5+3x$.

2. (a) Let $f(x) \in \mathbb{R}[x]$. Explain why $f(x) + \langle x^2+1\rangle$ can be "reduced" to $(a+bx) + \langle x^2+1\rangle$ where $a, b \in \mathbb{R}$. ← What are the remainders when dividing by x^2+1?

   (b) Describe all distinct elements of $\mathbb{R}[x]/\langle x^2+1\rangle$.

3. (a) In $\mathbb{R}[x]/\langle x^2+1\rangle$, explain why $x^2 + \langle x^2+1\rangle = -1 + \langle x^2+1\rangle$.

   Hint: $f(x) + \langle x^2+1\rangle = g(x) + \langle x^2+1\rangle$ if and only if... ?

   (b) Let $f(x) = 5x^4 + x^3 - 3x^2 + 4x - 3 \in \mathbb{R}[x]$ again. After completing part (a) above, Elizabeth wrote down the following calculation: ← We can treat x^2 and -1 to be the same *as coset representatives*.

$$\begin{aligned} f(x) + \langle x^2+1\rangle &= (5x^4 + x^3 - 3x^2 + 4x - 3) + \langle x^2+1\rangle \\ &= (5\cdot \boldsymbol{x^2}\cdot \boldsymbol{x^2} + \boldsymbol{x^2}\cdot x - 3\cdot \boldsymbol{x^2} + 4x - 3) + \langle x^2+1\rangle \\ &= (5\cdot \boldsymbol{(-1)}\cdot \boldsymbol{(-1)} + \boldsymbol{(-1)}\cdot x - 3\cdot \boldsymbol{(-1)} + 4x - 3) + \langle x^2+1\rangle \end{aligned}$$

   Complete her calculation to verify that $f(x) + \langle x^2+1\rangle = (5+3x) + \langle x^2+1\rangle$. ← Compare with Prob. #1. Which do you prefer?

4. Compute each product and reduce it using Elizabeth's method from Problem #3:

   (a) $((2+7x) + \langle x^2+1\rangle)\cdot((4+3x) + \langle x^2+1\rangle)$

   (b) $((-1+2x) + \langle x^2+1\rangle)\cdot((3+5x) + \langle x^2+1\rangle)$

5. Compute each product in $\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$, the *field* of complex numbers.

   Note: Recall that $i\cdot i = -1$.

   (a) $(2+7i)\cdot(4+3i)$

   (b) $(-1+2i)\cdot(3+5i)$

6. (a) Compare the description of the distinct elements of $\mathbb{R}[x]/\langle x^2+1\rangle$ that you gave in Problem #2(b) with $\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$.

   (b) Compare your answers in Problem #4 with those in Problem #5.

   (c) To which familiar ring is $\mathbb{R}[x]/\langle x^2+1\rangle$ isomorphic? Explain.

   (d) Is $\mathbb{R}[x]/\langle x^2+1\rangle$ a field? Why or why not? **Ans:** Yes, because $\mathbb{C}$ is.

7. Since $\mathbb{R}[x]/\langle x^2+1\rangle$ is a field, every nonzero element is a unit. Find $((3+4x)+\langle x^2+1\rangle)^{-1}$, i.e., find $(a+bx)+\langle x^2+1\rangle$ such that

$$((3+4x)+\langle x^2+1\rangle)\cdot((a+bx)+\langle x^2+1\rangle)=1+\langle x^2+1\rangle.$$

Note: If you're stuck, look ahead at Problem #8.

Ans: $(\frac{3}{25}-\frac{4}{25}x)+\langle x^2+1\rangle$.

8. (a) Compute the product $(3+4i)\cdot(3-4i)$.

   Ans to (a): 25.

   (b) Find real numbers a and b such that

$$\frac{1}{3+4i}\cdot\frac{3-4i}{3-4i}=a+bi.$$

   (c) Find $(3+4i)^{-1}$ in $\mathbb{C}$.

   Hint: You already found it.

   (d) Compare with Problem #7. What's going on here?

9. Find $((5+2x)+\langle x^2+1\rangle)^{-1}$ using the technique from Problem #8.

10. Define a function $\theta:\mathbb{C}\to\mathbb{R}[x]/\langle x^2+1\rangle$ where $\theta(a+bi)=(a+bx)+\langle x^2+1\rangle$ for all $a+bi\in\mathbb{C}$. Show that θ is a ring homomorphism.

    Note: It turns out that θ is a bijection. Thus, θ is in fact an *isomorphism*.

    ← Do you see why?

11. Prove that $\mathbb{C}$ is a field by showing that $(a+bi)^{-1}$ exists for all $a+bi\neq 0$.

12. Let F be a field and fix $g(x)\in F[x]$. Prove each of the following:

    (a) If $g(x)$ is factorable, then $F[x]/\langle g(x)\rangle$ is *not* a field.

    ← See if you can reproduce the proof from Day 33.

    (b) **(Optional challenge)** If $g(x)$ is unfactorable, then $F[x]/\langle g(x)\rangle$ is a field.

13. Let $f(x)=4x^5+5x^4+x^3+9x^2-3x+4\in\mathbb{R}[x]$.

    (a) Verify that $f(i)=0$ where $i=\sqrt{-1}\in\mathbb{C}$.

    (b) Verify that $f(-i)=0$ as well.

14. (a) Repeat Problem #13, this time with $f(x)=7x^{12}+7x^{10}-3x^5-3x^3+2x^2+2$.

    (b) **Prove:** Let $f(x)\in\mathbb{R}[x]$. If $i\in\mathbb{C}$ is a root of $f(x)$, then $-i$ is also a root of $f(x)$.

15. Consider the function $\varphi:\mathbb{R}[x]\to\mathbb{C}$ where $\varphi(f(x))=f(i)$ for all $f(x)\in\mathbb{R}[x]$. Prove that:

    (a) φ is a ring homomorphism.

    (b) $\ker\varphi=\langle x^2+1\rangle$. (**Hint:** Use Problem #14b.)

    (c) $\operatorname{im}\varphi=\mathbb{C}$.

    (d) What conclusion can you make using the First Isomorphism Theorem?