Abstract Algebra
Day 33 Class Work

For the problems below, fix $x^2 - 1 \in \mathbb{Z}_7[x]$ and define $\langle x^2 - 1 \rangle = \{(x^2 - 1) \cdot q(x) \mid q(x) \in \mathbb{Z}_7[x]\}$. ← i.e., the principal ideal generated by $x^2 - 1$.

1. List three elements of $\mathbb{Z}_7[x]$ that are contained in $\langle x^2 - 1 \rangle$.

2. Let $f(x) = 4x^5 + 2x^3 + 4x + 1 \in \mathbb{Z}_7[x]$ and recall that

$$f(x) = (x^2 - 1) \cdot (4x^3 + 6x) + (3x + 1).$$

← In $\mathbb{Z}_7[x]$, we have $10x + 1 = 3x + 1$.

   (a) Explain why $f(x)$ is *not* contained in $\langle x^2 - 1 \rangle$.

   (b) Using the result of the division algorithm above, find $g(x) \in \mathbb{Z}_7[x]$ of the smallest degree such that $f(x) + \langle x^2 - 1 \rangle = g(x) + \langle x^2 - 1 \rangle$. **Ans:** $g(x) = 3x + 1$.

   (c) In part (b), describe how $f(x)$ and $g(x)$ are related. (**Hint:** Think $f(x) - g(x)$.)

3. Let $f(x) = 2x^9 + 5x^7 + 4x^3 + 3 \in \mathbb{Z}_7[x]$.

   (a) Find $g(x) \in \mathbb{Z}_7[x]$ of the smallest degree such that $f(x) + \langle x^2 - 1 \rangle = g(x) + \langle x^2 - 1 \rangle$. The following Mathematica code should help:

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In[16]:= f = 2 x^9 + 5 x^7 + 4 x^3 + 3

In[17]:= PolynomialMod[PolynomialQuotientRemainder[f, x^2 - 1, x], 7]

Out[17]= {2 x^7 + 4 x, 4 x + 3}
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   (b) In part (a), describe how $f(x)$ and $g(x)$ are related.

4. (a) Let $f(x) \in \mathbb{Z}_7[x]$. Explain why $f(x) + \langle x^2 - 1 \rangle$ can be "reduced" to $(ax + b) + \langle x^2 - 1 \rangle$ where a, $b \in \mathbb{Z}_7$.

   (b) Describe all distinct elements of $\mathbb{Z}_7[x]/\langle x^2 - 1 \rangle$. How many are there? **Ans:** 49 cosets.

5. (a) In $\mathbb{Z}_7[x]/\langle x^2 - 1 \rangle$, explain why $x^2 + \langle x^2 - 1 \rangle = 1 + \langle x^2 - 1 \rangle$.

   Hint: $f(x) + \langle x^2 - 1 \rangle = g(x) + \langle x^2 - 1 \rangle$ if and only if… ?

   (b) Let $f(x) = 4x^5 + 2x^3 + 4x + 1 \in \mathbb{Z}_7[x]$ again. After completing part (a) above, Elizabeth wrote down the following calculation:

$$\begin{aligned} f(x) + \langle x^2 - 1 \rangle &= (4x^5 + 2x^3 + 4x + 1) + \langle x^2 - 1 \rangle \\ &= (4 \cdot \boldsymbol{x^2} \cdot \boldsymbol{x^2} \cdot x + 2 \cdot \boldsymbol{x^2} \cdot x + 4x + 1) + \langle x^2 - 1 \rangle \\ &= (4 \cdot \mathbf{1} \cdot \mathbf{1} \cdot x + 2 \cdot \mathbf{1} \cdot x + 4x + 1) + \langle x^2 - 1 \rangle \end{aligned}$$

   Complete her calculation to verify that $f(x) + \langle x^2 - 1 \rangle = (3x + 1) + \langle x^2 - 1 \rangle$. ← Compare with Prob. #2. Which do you prefer?

   (c) Anita says, "I see what she did in part (b). When dealing with coset representatives, we can treat x^2 and 1 to be the same." What might Anita mean?

   (d) Apply Elizabeth's method to Problem #3. Did you get the same result?

6. Working on Problem #4, Anita says:

   > "I found 49 cosets of the form $(ax + b) + \langle x^2 - 1 \rangle$. But how do I know for sure that they're all distinct? Why can't, for instance, $(5x + 3) + \langle x^2 - 1 \rangle$ and $(3x + 6) + \langle x^2 - 1 \rangle$ be equal?"

   How would you respond to her?

7. (a) Find a zero divisor in $\mathbb{Z}_7[x]/\langle x^2-1\rangle$. (**Hint:** x^2-1 factors. How does that help?)

   (b) Is $\mathbb{Z}_7[x]/\langle x^2-1\rangle$ a field? Why or why not? **Ans:** No. (Why not?)

8. Consider the elements $(4x+3)+\langle x^2-1\rangle$ and $(4x+2)+\langle x^2-1\rangle$ in $\mathbb{Z}_7[x]/\langle x^2-1\rangle$. Determine if each is a unit or a zero divisor. Explain how you know.

   Hint: Compute $((4x+3)+\langle x^2-1\rangle)\cdot((ax+b)+\langle x^2-1\rangle)$. How can it be reduced?

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9. Explore the quotient ring $\mathbb{Z}_7[x]/\langle x^2+1\rangle$, where $\langle x^2+1\rangle = \{(x^2+1)\cdot q(x) \mid q(x)\in\mathbb{Z}_7[x]\}$.

   (a) Describe all distinct elements of $\mathbb{Z}_7[x]/\langle x^2+1\rangle$. How many are there?

   (b) Consider the elements $(4x+3)+\langle x^2+1\rangle$ and $(4x+2)+\langle x^2+1\rangle$ in $\mathbb{Z}_7[x]/\langle x^2+1\rangle$. Determine if each is a unit or a zero divisor. Explain how you know.

   (c) Either find a zero divisor in $\mathbb{Z}_7[x]/\langle x^2+1\rangle$ or explain why one doesn't exist.