Abstract Algebra
Day 32 Class Work

For the problems below, fix $x^2 \in \mathbb{Z}_3[x]$ and define $\langle x^2 \rangle = \{x^2 \cdot q(x) \mid q(x) \in \mathbb{Z}_3[x]\}$. ← i.e., $\langle x^2 \rangle$ is the principal ideal generated by x^2.

1. (a) Explain why $2x^7 + x^5 + 2x^4$ is contained in $\langle x^2 \rangle$.

   (b) Explain why $x^6 + 2x$ is *not* contained in $\langle x^2 \rangle$.

   (c) List three more elements of $\mathbb{Z}_3[x]$ that are contained in $\langle x^2 \rangle$.

   (d) List three more elements of $\mathbb{Z}_3[x]$ that are *not* contained in $\langle x^2 \rangle$.

2. Let $\alpha(x), \beta(x) \in \mathbb{Z}_3[x]$ where $\alpha(x) = 2x^7 + x^5 + 2x^4 + \mathbf{2x + 1}$ and $\beta(x) = 2x + 1$. Explain why the cosets $\alpha(x) + \langle x^2 \rangle$ and $\beta(x) + \langle x^2 \rangle$ are equal, i.e., $\alpha(x) + \langle x^2 \rangle = \beta(x) + \langle x^2 \rangle$.

   Hint: In $\mathbb{Z}/5\mathbb{Z}$, $a + 5\mathbb{Z} = b + 5\mathbb{Z}$ if and only if... (What about in $\mathbb{Z}_3[x]/\langle x^2 \rangle$?)

3. (a) Let $f(x) = x^5 + 2x^2 + x + 2 \in \mathbb{Z}_3[x]$. Find $g(x) \in \mathbb{Z}_3[x]$ of smallest degree such that ← $\deg g(x) = 1$.

   $$f(x) + \langle x^2 \rangle = g(x) + \langle x^2 \rangle.$$

   (b) Repeat part (a), this time with $f(x) = 2x^9 + x^6 + 2x^3 + 1 \in \mathbb{Z}_3[x]$.

4. (a) Find all distinct elements of $\mathbb{Z}_3[x]/\langle x^2 \rangle$, i.e., distinct cosets of $\langle x^2 \rangle$ in $\mathbb{Z}_3[x]$. ← There are 9 cosets.

   (b) Find the additive identity and the multiplicative identity of $\mathbb{Z}_3[x]/\langle x^2 \rangle$.

   (c) Elizabeth says: "I found 9 cosets of the form $(ax + b) + \langle x^2 \rangle$. But how do I know for sure that they're all distinct? Why can't, for instance, $(2x+1)+\langle x^2 \rangle = (x+2)+\langle x^2 \rangle$?" How would you respond to Elizabeth?

5. (a) Compute $((x+1)+\langle x^2 \rangle)\cdot((2x+1)+\langle x^2 \rangle)$ and verify that the product equals $1+\langle x^2 \rangle$.

   (b) Anita says, "Our work in part (a) shows that $(x + 1) + \langle x^2 \rangle$ and $(2x + 1) + \langle x^2 \rangle$ are units." What might she mean?

6. The quotient ring $\mathbb{Z}_3[x]/\langle x^2 \rangle$ is finite, so every nonzero element must be either a unit or a zero divisor (i.e., it cannot be neither). For each nonzero element $a(x) + \langle x^2 \rangle$, determine if it's a unit or a zero divisor. Moreover, find a nonzero $b(x) + \langle x^2 \rangle$ such that...

   - For a unit: $(a(x) + \langle x^2 \rangle) \cdot (b(x) + \langle x^2 \rangle) = 1 + \langle x^2 \rangle$.
   - For a zero divisor: $(a(x) + \langle x^2 \rangle) \cdot (b(x) + \langle x^2 \rangle) = 0 + \langle x^2 \rangle$.

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7. Consider the quotient ring $\mathbb{Z}_2[x]/\langle x^2 \rangle$. (**Be careful:** The coefficient ring is $\mathbb{Z}_2$, not $\mathbb{Z}_3$.)

   (a) Find all distinct elements of $\mathbb{Z}_2[x]/\langle x^2 \rangle$, i.e., cosets $f(x) + \langle x^2 \rangle$ where $f(x) \in \mathbb{Z}_2[x]$. ← There are 4 cosets.

   (b) Construct the addition and multiplication tables for $\mathbb{Z}_2[x]/\langle x^2 \rangle$.

   (c) Is $\mathbb{Z}_2[x]/\langle x^2 \rangle$ a field? Why or why not? **Ans:** No. (Why not?)

8. (a) How many distinct elements does $\mathbb{Z}_5[x]/\langle x^2 \rangle$ contain?

   (b) How about $\mathbb{Z}_7[x]/\langle x^2 \rangle$?

(c) How about $\mathbb{Z}_{11}[x]/\langle x^2 \rangle$?

(d) How about $\mathbb{Z}_{101}[x]/\langle x^2 \rangle$?

(e) How about $\mathbb{Z}_p[x]/\langle x^2 \rangle$, where p is prime? ← There are p^2 cosets.

9. (a) Repeat Problem #8, but replace $\langle x^2 \rangle$ with $\langle x^3 \rangle$.

   (b) Repeat Problem #8, but replace $\langle x^2 \rangle$ with $\langle x^n \rangle$, where n is a positive integer.

10. Explore the quotient ring $\mathbb{Z}_3[x]/\langle x^2+1 \rangle$, where $\langle x^2+1 \rangle = \{(x^2+1) \cdot q(x) \mid q(x) \in \mathbb{Z}_3[x]\}$.

   (a) Find all distinct elements of $\mathbb{Z}_3[x]/\langle x^2+1 \rangle$, i.e., distinct cosets of $\langle x^2+1 \rangle$ in $\mathbb{Z}_3[x]$. ← There are 9 cosets.

   (b) For each nonzero element $a(x) + \langle x^2+1 \rangle$, determine if it's a unit or a zero divisor.

   (c) Moreover, find a nonzero $b(x) + \langle x^2+1 \rangle$ such that...

   - For a unit: $(a(x) + \langle x^2+1 \rangle) \cdot (b(x) + \langle x^2+1 \rangle) = 1 + \langle x^2+1 \rangle$.
   - For a zero divisor: $(a(x) + \langle x^2+1 \rangle) \cdot (b(x) + \langle x^2+1 \rangle) = 0 + \langle x^2+1 \rangle$.