Abstract Algebra
Day 31 Class Work Solutions

Below, we will look at the following functions:

- $\theta : \mathbb{R}[x] \to \mathbb{R}$ where $\theta(f(x)) = f(2)$ for all $f(x) \in \mathbb{R}[x]$. ← We've seen that θ is a ring homomorphism.
- $\varphi : \mathbb{Z} \to \mathbb{Z}_5$ where $\varphi(a) = a \pmod 5$ for all $a \in \mathbb{Z}$.
- $\lambda : \mathbb{Z}_{10} \to \mathbb{Z}_{30}$ where $\lambda(a) = 6a$ for all $a \in \mathbb{Z}_{10}$.

---

1. Consider the function φ.

   (a) Compute $\varphi(26 + 17)$ and $\varphi(26) + \varphi(17)$ and verify that they're equal.

   Solution. We have $\varphi(26 + 17) = \varphi(43) = 43 \pmod 5 = 3 \pmod 5$, and

   $$\varphi(26) + \varphi(17) = 26 \pmod 5 + 17 \pmod 5 = 1 \pmod 5 + 2 \pmod 5 = 3 \pmod 5.$$

   Thus, the two expressions are equal.

   (b) Compute $\varphi(26 \cdot 17)$ and $\varphi(26) \cdot \varphi(17)$ and verify that they're equal. ← Which one is easier?

   Solution. We have $\varphi(26 \cdot 17) = \varphi(442) = 2 \pmod 5$, and

   $$\varphi(26) \cdot \varphi(17) = 26 \pmod 5 \cdot 17 \pmod 5 = 1 \pmod 5 \cdot 2 \pmod 5 = 2 \pmod 5.$$

   Thus, the two expressions are equal.

   Note: The above relationships always hold, so that φ is a ring homomorphism. ← Can you *prove* them?

2. Now consider the function λ.

   (a) Verify that $\lambda(7 + 4) = \lambda(7) + \lambda(4)$ and that $\lambda(7 \cdot 4) = \lambda(7) \cdot \lambda(4)$.

   Solution. We have $\lambda(7 + 4) = \lambda(1) = 6 \cdot 1 = 6$, and ← Note that $11 = 1$ in $\mathbb{Z}_{10}$.

   $$\lambda(7) + \lambda(4) = 6 \cdot 7 + 6 \cdot 4 = 12 + 24 = 6.$$

   We also have $\lambda(7 \cdot 4) = \lambda(8) = 6 \cdot 8 = 18$, and

   $$\lambda(7) \cdot \lambda(4) = (6 \cdot 7) \cdot (6 \cdot 4) = 12 \cdot 24 = 288 = 18.$$

   (b) Show that $\lambda(a + b) = \lambda(a) + \lambda(b)$ and $\lambda(a \cdot b) = \lambda(a) \cdot \lambda(b)$ for all $a, b \in \mathbb{Z}_{10}$. ← Therefore, λ is a ring homomorphism.

   Solution. Let $a, b \in \mathbb{Z}_{10}$. We have $\lambda(a + b) = 6(a + b) = 6a + 6b = \lambda(a) + \lambda(b)$. Moreover, we have $\lambda(a \cdot b) = 6(ab)$. We also have $\lambda(a) \cdot \lambda(b) = 6a \cdot 6b = 36(ab) = 6(ab)$, since $36 = 6$ in $\mathbb{Z}_{30}$. Therefore, $\lambda(a \cdot b) = \lambda(a) \cdot \lambda(b)$.

3. Define the *kernel* of θ as follows: $\ker \theta = \{f(x) \in \mathbb{R}[x] \mid \theta(f(x)) = 0\}$.

   Note: The kernel is the set of elements of the domain that map to the *additive* identity.

   (a) Determine if $f(x) = x^2 + x - 6$ and $g(x) = x^3 - 7$ are in $\ker \theta$.

   Solution. We have $\theta(f(x)) = f(2) = 2^2 + 2 - 6 = 0$ and $\theta(g(x)) = g(2) = 2^3 - 7 = 1$. Thus, $f(x) \in \ker \theta$, but $g(x) \notin \ker \theta$.

   (b) Describe all polynomials that are in $\ker \theta$. ← **Ans:** $f(x) \in \ker \theta$ means $f(x)$ has $x - 2$ as a factor.

   Solution. $f(x)$ is in $\ker \theta$ when $\theta(f(x)) = 0$, i.e., when $f(2) = 0$. These are precisely the polynomials that have $x - 2$ as a factor, such as $x^2 - 4$ and $x^2 + x - 6$.

(c) Compute the kernel of φ and λ.

Solution. We have $\ker\varphi = 5\mathbb{Z}$ and $\ker\lambda = \{0, 5\}$.

4. Consider again the function φ. You should've found that $\ker\varphi = 5\mathbb{Z}$.

(a) Verify that $5\mathbb{Z}$ is an additive sub*group* of the domain $\mathbb{Z}$.

Solution. We must show that $5\mathbb{Z}$ is closed under addition, that $5\mathbb{Z}$ contains the additive identity 0, and that if $a \in 5\mathbb{Z}$ then $-a \in 5\mathbb{Z}$. I'll leave the details up to you!

(b) Explain why $5\mathbb{Z}$ is **not** a sub*ring* of the domain $\mathbb{Z}$.

← Which important ring element is $5\mathbb{Z}$ missing?

Solution. $5\mathbb{Z}$ does *not* contain the multiplicative identity 1.

(c) Show that $5\mathbb{Z}$ satisfies the *product absorption* property:

If $r \in \mathbb{Z}$ (the domain) and $a \in 5\mathbb{Z}$, then $r \cdot a \in 5\mathbb{Z}$.

PROOF. Suppose $r \in \mathbb{Z}$ and $a \in 5\mathbb{Z}$ so that $a = 5n$ for some $n \in \mathbb{Z}$. Then $r \cdot a = r \cdot 5n = 5(rn) \in 5\mathbb{Z}$. ■

5. Consider the function $\theta : \mathbb{R}[x] \to \mathbb{R}$, and let $K = \ker\theta$.

(a) **Discuss in your group:** Anita says, "Well, θ is a homomorphism of additive groups, so K should be a sub*group* of the domain $\mathbb{R}[x]$." What might she mean?

Recall: Every ring is an additive group.

Solution. Every ring is a group under addition. Thus, $\mathbb{R}[x]$ and $\mathbb{R}$ are additive groups. Moreover, θ preserves addition, i.e., $\theta(f(x) + g(x)) = \theta(f(x)) + \theta(g(x))$ for all $f(x), g(x) \in \mathbb{R}[x]$. Hence, we may view θ as a *group* homomorphism, and we know from group theory that $K = \ker\theta$ is an (additive) subgroup of the domain $\mathbb{R}[x]$.

(b) Show that K satisfies the product absorption property:

If $f(x) \in \mathbb{R}[x]$ (the domain) and $k(x) \in K$, then $f(x) \cdot k(x) \in K$.

PROOF. Let $f(x) \in \mathbb{R}[x]$ and $k(x) \in K$ so that $\theta(k(x)) = 0$, i.e., $k(2) = 0$. We have $\theta(f(x) \cdot k(x)) = \theta(f(x)) \cdot \theta(k(x)) = f(2) \cdot k(2) = f(2) \cdot 0 = 0$. So, $f(x) \cdot k(x) \in K$. ■

6. In Problem #3(c), you should've found that $\ker\lambda = \{0, 5\}$. Verify that $\ker\lambda$ also satisfies product absorption:

If $r \in \mathbb{Z}_{10}$ (the domain) and $a \in \{0, 5\}$, then $r \cdot a \in \{0, 5\}$.

Solution. Suppose $r \in \mathbb{Z}_{10}$ and $a \in \{0, 5\}$. There are two cases to consider: $a = 0$ or $a = 5$. If $a = 0$, then $r \cdot a = 0 \in \{0, 5\}$. If $a = 5$, then $r \cdot a = 0$ for $r = 0, 2, 4, 6, 8$; and $r \cdot a = 5$ for $r = 1, 3, 5, 7, 9$. Thus, in either case, $r \cdot a \in \{0, 5\}$.

7. Let $\theta : R \to S$ be a ring homomorphism with $K = \ker\theta = \{r \in R \mid \theta(r) = 0_S\}$. Prove that...

← Here, 0_S refers to the additive identity of S.

(a) K is an additive subgroup of R. (Recreate the proof from group theory.)

Note: The operation here addition. When referring to the inverse of $k \in K$, for instance, consider the *additive* inverse $-k$ (instead of the multiplicative inverse k^{-1}).

PROOF. Let $a, b \in K$ so that $\theta(a) = 0_S$ and $\theta(b) = 0_S$. Since θ is operation preserving, $\theta(a + b) = \theta(a) + \theta(b) = 0_S + 0_S = 0_S$. Thus, $a + b \in K$ and so K is closed under addition. We have $\theta(0_R) = 0_S$, and thus $0_R \in K$. Finally, note that $\theta(-a) = -\theta(a) = -0_S = 0_S$, which shows $-a \in K$. Hence, K is a subgroup of R. ■

(b) If $r \in R$ and $a \in K$, then $r \cdot a \in K$.

Note: In other words, $K = \ker \theta$ satisfies product absorption.

PROOF. Let $r \in R$ and $k \in K$. Then

$$\theta(r \cdot k) = \theta(r) \cdot \theta(k) = \theta(r) \cdot 0_S = 0_S.$$

Hence, $r \cdot k \in K$. ■

8. Define the *image* of θ as follows: $\operatorname{im} \theta = \{\theta(f(x)) \mid f(x) \in \mathbb{R}[x]\}$.

(a) Explain why $\operatorname{im} \theta = \mathbb{R}$.

Solution. Given $a \in \mathbb{R}$, let $f(x) = (x-2) + a \in \mathbb{R}[x]$. Then, $\theta(f(x)) = f(2) = (2-2) + a = a$. Thus, $a \in \operatorname{im} \theta$ and so $\operatorname{im} \theta$ is all of the codomain $\mathbb{R}$.

(b) Compute the image of φ and λ.

Solution. We have $\operatorname{im} \varphi = \mathbb{Z}_5$ and $\operatorname{im} \lambda = \{0, 6, 12, 18, 24\}$.

9. (a) Is θ one-to-one? Is it onto?

Ans to (a): No and Yes.

Solution. θ is not one-to-one. As a counterexample, suppose $f(x) = x^2 - 4$ and $g(x) = x^2 + x - 6$. Then $f(x) \neq g(x)$, but $\theta(f(x)) = f(2) = 0$ and $\theta(g(x)) = g(2) = 0$ so that $\theta(f(x)) = \theta(g(x))$.

In Problem #8(a), we saw that $\operatorname{im} \theta$ equals $\mathbb{R}$. Thus, θ is onto, because every element in $\mathbb{R}$ (the codomain) gets "hit" by the function θ.

(b) Answer the same questions for φ and λ.

Solution. φ is *not* one-to-one. As a counterexample, suppose $a = 7$ and $b = 12$. Then $a \neq b$, but $\varphi(a) = \varphi(b)$. φ is onto—I'll leave the explanation up to you!

λ is *not* one-to-one, as $\lambda(0) = \lambda(5)$. Since the codomain $\mathbb{Z}_{30}$ has more elements than the domain $\mathbb{Z}_{10}$, it isn't possible for λ (or any function from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{30}$) to be onto.

10. Consider the function $\theta : \mathbb{Z}_2[x] \to \mathbb{Z}_2[x]$ where $\theta(f(x)) = f(x)^2$ for all $f(x) \in \mathbb{Z}_2[x]$.

(a) Let $f(x), g(x) \in \mathbb{Z}_2[x]$ where $f(x) = x^2 + 1$ and $g(x) = x + 1$. Compute $\theta(f(x) + g(x))$ and $\theta(f(x)) + \theta(g(x))$ and verify that they're equal.

← Remember that the coefficients are in $\mathbb{Z}_2$.

(b) For the same $f(x)$, $g(x)$ from part (a), compute $\theta(f(x) \cdot g(x))$ and $\theta(f(x)) \cdot \theta(g(x))$ and verify that they're equal.

(c) Prove that θ is a ring homomorphism.

Solution. See Example 31.8 in the textbook for details.

11. Consider the following functions from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{10}$.

- $\varphi_0 : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ where $\varphi_0(x) = 0 \cdot x$ for all $x \in \mathbb{Z}_{10}$.
- $\varphi_1 : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ where $\varphi_1(x) = 1 \cdot x$ for all $x \in \mathbb{Z}_{10}$.
- $\varphi_2 : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ where $\varphi_2(x) = 2 \cdot x$ for all $x \in \mathbb{Z}_{10}$.
- $\varphi_3 : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ where $\varphi_3(x) = 3 \cdot x$ for all $x \in \mathbb{Z}_{10}$.
- (...and so on, until...)
- $\varphi_9 : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ where $\varphi_9(x) = 9 \cdot x$ for all $x \in \mathbb{Z}_{10}$.

(a) Explain why φ_5 is a ring homomorphism, but φ_4 is not.

(b) Find all $k \in \mathbb{Z}_{10}$ for which φ_k is a ring homomorphism.

12. (a) Repeat Problem #11, but consider functions from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{12}$.
    (b) Same as part (a), but from $\mathbb{Z}_7$ to $\mathbb{Z}_7$; from $\mathbb{Z}_{15}$ to $\mathbb{Z}_{15}$.
    (c) Consider the function $\varphi_k : \mathbb{Z}_n \to \mathbb{Z}_n$ where $\varphi_k(x) = k \cdot x$ for all $x \in \mathbb{Z}_n$. Find the values of k for which φ_k a ring homomorphism. Explain how you know.

13. **Prove:** Every ring homomorphism $\theta : \mathbb{Z}_n \to \mathbb{Z}_n$ has the form $\theta(x) = k \cdot x$ where $k^2 = k$. ← i.e., k is an idempotent.