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Abstract Algebra
Day 31 Class Work

Below, we will look at the following functions:

- $\theta : \mathbb{R}[x] \to \mathbb{R}$ where $\theta(f(x)) = f(2)$ for all $f(x) \in \mathbb{R}[x]$. ← We've seen that θ is a ring homomorphism.
- $\varphi : \mathbb{Z} \to \mathbb{Z}_5$ where $\varphi(a) = a \pmod 5$ for all $a \in \mathbb{Z}$.
- $\lambda : \mathbb{Z}_{10} \to \mathbb{Z}_{30}$ where $\lambda(a) = 6a$ for all $a \in \mathbb{Z}_{10}$.

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1. Consider the function φ.

   (a) Compute $\varphi(26 + 17)$ and $\varphi(26) + \varphi(17)$ and verify that they're equal.

   (b) Compute $\varphi(26 \cdot 17)$ and $\varphi(26) \cdot \varphi(17)$ and verify that they're equal. ← Which one is easier?

   Note: The above relationships always hold, so that φ is a ring homomorphism. ← Can you *prove* them?

2. Now consider the function λ.

   (a) Verify that $\lambda(7 + 4) = \lambda(7) + \lambda(4)$ and that $\lambda(7 \cdot 4) = \lambda(7) \cdot \lambda(4)$.

   (b) Show that $\lambda(a + b) = \lambda(a) + \lambda(b)$ and $\lambda(a \cdot b) = \lambda(a) \cdot \lambda(b)$ for all $a, b \in \mathbb{Z}_{10}$. ← Therefore, λ is a ring homomorphism.

3. Define the *kernel* of θ as follows: $\ker\theta = \{f(x) \in \mathbb{R}[x] \mid \theta(f(x)) = 0\}$.

   Note: The kernel is the set of elements of the domain that map to the *additive* identity.

   (a) Determine if $f(x) = x^2 + x - 6$ and $g(x) = x^3 - 7$ are in $\ker\theta$.

   (b) Describe all polynomials that are in $\ker\theta$. **Ans:** $f(x) \in \ker\theta$ means $f(x)$ has $x - 2$ as a factor.

   (c) Compute the kernel of φ and λ.

4. Consider again the function φ. You should've found that $\ker\varphi = 5\mathbb{Z}$.

   (a) Verify that $5\mathbb{Z}$ is an additive sub*group* of the domain $\mathbb{Z}$.

   (b) Explain why $5\mathbb{Z}$ is **not** a sub*ring* of the domain $\mathbb{Z}$. ← Which important ring element is $5\mathbb{Z}$ missing?

   (c) Show that $5\mathbb{Z}$ satisfies the *product absorption* property:

   If $r \in \mathbb{Z}$ (the domain) and $a \in 5\mathbb{Z}$, then $r \cdot a \in 5\mathbb{Z}$.

5. Consider the function $\theta : \mathbb{R}[x] \to \mathbb{R}$, and let $K = \ker\theta$.

   (a) **Discuss in your group:** Anita says, "Well, θ is a homomorphism of additive groups, so K should be a sub*group* of the domain $\mathbb{R}[x]$." What might she mean? **Recall:** Every ring is an additive group.

   (b) Show that K satisfies the product absorption property:

   If $f(x) \in \mathbb{R}[x]$ (the domain) and $k(x) \in K$, then $f(x) \cdot k(x) \in K$.

6. In Problem #3(c), you should've found that $\ker\lambda = \{0, 5\}$. Verify that $\ker\lambda$ also satisfies product absorption:

   If $r \in \mathbb{Z}_{10}$ (the domain) and $a \in \{0, 5\}$, then $r \cdot a \in \{0, 5\}$.

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7. Let $\theta : R \to S$ be a ring homomorphism with $K = \ker \theta = \{r \in R \mid \theta(r) = 0_S\}$. Prove that...

   ← Here, 0_S refers to the additive identity of S.

   (a) K is an additive subgroup of R. (Recreate the proof from group theory.)

   Note: The operation here addition. When referring to the inverse of $k \in K$, for instance, consider the *additive* inverse $-k$ (instead of the multiplicative inverse k^{-1}).

   (b) If $r \in R$ and $a \in K$, then $r \cdot a \in K$.

   Note: In other words, $K = \ker \theta$ satisfies product absorption.

8. Define the *image* of θ as follows: $\operatorname{im} \theta = \{\theta(f(x)) \mid f(x) \in \mathbb{R}[x]\}$.

   (a) Explain why $\operatorname{im} \theta = \mathbb{R}$.

   (b) Compute the image of φ and λ.

9. (a) Is θ one-to-one? Is it onto?

   Ans to (a): No and Yes.

   (b) Answer the same questions for φ and λ.

10. Consider the function $\theta : \mathbb{Z}_2[x] \to \mathbb{Z}_2[x]$ where $\theta(f(x)) = f(x)^2$ for all $f(x) \in \mathbb{Z}_2[x]$.

   (a) Let $f(x), g(x) \in \mathbb{Z}_2[x]$ where $f(x) = x^2+1$ and $g(x) = x+1$. Compute $\theta(f(x)+g(x))$ and $\theta(f(x)) + \theta(g(x))$ and verify that they're equal.

   ← Remember that the coefficients are in $\mathbb{Z}_2$.

   (b) For the same $f(x)$, $g(x)$ from part (a), compute $\theta(f(x) \cdot g(x))$ and $\theta(f(x)) \cdot \theta(g(x))$ and verify that they're equal.

   (c) Prove that θ is a ring homomorphism.

11. Consider the following functions from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{10}$.

   - $\varphi_0 : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ where $\varphi_0(x) = 0 \cdot x$ for all $x \in \mathbb{Z}_{10}$.
   - $\varphi_1 : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ where $\varphi_1(x) = 1 \cdot x$ for all $x \in \mathbb{Z}_{10}$.
   - $\varphi_2 : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ where $\varphi_2(x) = 2 \cdot x$ for all $x \in \mathbb{Z}_{10}$.
   - $\varphi_3 : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ where $\varphi_3(x) = 3 \cdot x$ for all $x \in \mathbb{Z}_{10}$.
   - (...and so on, until...)
   - $\varphi_9 : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ where $\varphi_9(x) = 9 \cdot x$ for all $x \in \mathbb{Z}_{10}$.

   (a) Explain why φ_5 is a ring homomorphism, but φ_4 is not.

   (b) Find all $k \in \mathbb{Z}_{10}$ for which φ_k is a ring homomorphism.

12. (a) Repeat Problem #11, but consider functions from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{12}$.

   (b) Same as part (a), but from $\mathbb{Z}_7$ to $\mathbb{Z}_7$; from $\mathbb{Z}_{15}$ to $\mathbb{Z}_{15}$.

   (c) Consider the function $\varphi_k : \mathbb{Z}_n \to \mathbb{Z}_n$ where $\varphi_k(x) = k \cdot x$ for all $x \in \mathbb{Z}_n$. Find the values of k for which φ_k a ring homomorphism. Explain how you know.

13. **Prove:** Every ring homomorphism $\theta : \mathbb{Z}_n \to \mathbb{Z}_n$ has the form $\theta(x) = k \cdot x$ where $k^2 = k$.

   ← i.e., k is an idempotent.