Abstract Algebra
Day 29 Class Work

1. Consider $f(x) = 4x^3 + 5x^2 + 2$ and $g(x) = 3x^2 + 5$ in $\mathbb{Z}_7[x]$.

   (a) Use long division to compute the quotient $q(x)$ and remainder $r(x)$ when dividing $f(x)$ by $g(x)$. Keep in mind that the coefficients are in $\mathbb{Z}_7$. **Ans:** $q(x) = 6x + 4$, $r(x) = 5x + 3$.

   (b) Verify that your result in part (a) satisfies the division algorithm for polynomials.

2. Consider $f(x) \in \mathbb{R}[x]$ where

$$f(x) = (x - 2) \cdot (7432x^{3914} - 652x^{1842} + 37x^{953} + 6x^{75} - 4321x^{59} + 1023).$$

   Explain why $f(2) = 0$.

3. Let $f(x) \in \mathbb{R}[x]$ and suppose $x - 2$ is a factor of $f(x)$, i.e., $f(x) = (x - 2) \cdot q(x)$ for some $q(x) \in \mathbb{R}[x]$. Explain why $f(2) = 0$.

4. Let $f(x) = 4x^3 - 9x^2 + 5x - 6 \in \mathbb{R}[x]$.

   (a) Compute $f(2)$ and verify that $f(2) = 0$.

   (b) What does your result in part (a) say about how $f(x)$ factors?

   (c) Use long division to compute the quotient $q(x)$ and remainder $r(x)$ when dividing $f(x)$ by $x - 2$. Explain how this confirms your answer from part (b). ← What *should* $r(x)$ be?

5. **Prove:** Let $f(x) \in \mathbb{R}[x]$. If $f(2) = 0$, then $f(x) = (x - 2) \cdot q(x)$ for some $q(x) \in \mathbb{R}[x]$. ← Converse of Problem #3.

   Hint: Use the division algorithm for polynomials to write $f(x) = (x - 2) \cdot q(x) + r(x)$. What can you say about the remainder $r(x)$?

6. Consider $f(x) = 5x^{672} + 2x^{359} + 4x^{101} + x^{77} + 3x^{23} + 6$ in $\mathbb{Z}_7[x]$.

   (a) Show that $x - 1$ is a factor of $f(x)$. **Hint:** Compute $f(1)$.

   (b) Show that $x + 1$ is *not* a factor of $f(x)$.

7. (a) Find the remainder when $f(x) = 5x^{451} + 11x^{274} + 1$ is divided by $x - 1$ in $\mathbb{Z}_{13}[x]$.

   Hint: Use the division algorithm for polynomials.

   (b) Find the remainder when x^{50} is divided by $x + 2$ in $\mathbb{Z}_5[x]$. **Ans:** Remainder = 4.

8. (a) Find $p(x)$, $q(x) \in \mathbb{Z}_{10}[x]$, both with degree 1, such that $p(x) \cdot q(x) = x + 7$. ← Bonus fun!

   (b) What if $p(x)$ and $q(x)$ must each have degree greater than 1? Do such polynomials exist in $\mathbb{Z}_{10}[x]$? If so, find them. If not, explain why not.

9. Let $f(x) = x^3$ and $g(x) = 2x$ in $\mathbb{Z}[x]$.

   (a) Explain why there does *not* exist $q(x)$, $r(x) \in \mathbb{Z}[x]$ such that $x^3 = 2x \cdot q(x) + r(x)$, with either $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

   (b) Does your answer in part (a) contradict the division algorithm for polynomials? **Ans:** No. (Why not?)

Definition. A ring element r is said to be *nilpotent* if $r^n = 0$ for some positive integer n.

← In any ring, 0 is nilpotent, since $0^1 = 0$.

Example: $3 \in \mathbb{Z}_{81}$ is nilpotent, because $3^4 = 0$ in $\mathbb{Z}_{81}$.

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10. (a) Find all nilpotent elements of $\mathbb{Z}_9$.

    Ans: 0, 3, and 6.

    (b) Find all nilpotent elements of $\mathbb{Z}_{10}$; of $\mathbb{Z}_{12}$; of $\mathbb{Z}_{36}$.

    (c) Any conjectures about which $\mathbb{Z}_m$ has nonzero nilpotent elements?

11. In the polynomial ring $\mathbb{Z}_4[x]$, 1 and 3 are units and 0 and 2 are nilpotent elements.

    ← Do you see why?

    (a) In $\mathbb{Z}_4[x]$, find five more units and five more nilpotent elements.

    (b) Explain why $\mathbb{Z}_4[x]$ has infinitely many units and infinitely many nilpotent elements.