Abstract Algebra
Day 28 Class Work

1. Consider the polynomials $f(x) = 3x^4 - 7x^2 + 4$ and $g(x) = 4x^2 + 1$ in $\mathbb{Z}[x]$.

   (a) Find the sum $f(x) + g(x)$ and the product $f(x) \cdot g(x)$.

   (b) Verify that the sum and product that you found are also in $\mathbb{Z}[x]$.

   (c) Convince yourself that $\mathbb{Z}[x]$ is closed under addition and multiplication. ← No proof required.

2. (a) Verify that $\mathbb{Z}[x]$ is a *commutative* ring, focusing on the following questions. You can "skim" through the other ring properties.

   - What are the additive and multiplicative identities of $\mathbb{Z}[x]$?
   - What's the additive inverse of, say, $f(x) = 3x^4 - 7x^2 + 4$?
   - Do $f(x) + g(x) = g(x) + f(x)$ and $f(x) \cdot g(x) = g(x) \cdot f(x)$?

   Ans: Still 0 and 1, which are *constant* polynomials.

   (b) Anita says, "$\mathbb{Z}$ is a subring of $\mathbb{Z}[x]$." Do you agree or disagree with her? Explain.

   (c) Explain why x does *not* have a multiplicative inverse in $\mathbb{Z}[x]$.

   (d) Elizabeth says, "$\mathbb{Z}[x]$ is *not* a field, and neither is $\mathbb{R}[x]$." What do you think?

   Recall: In a *field*, every nonzero element is a unit.

3. Consider the polynomials $f(x) = 3x^{15} + 4x^3 + 2$ and $g(x) = 6x^8 + 5x + 3$.

   (a) In $\mathbb{Z}[x]$, compute the degrees of $f(x)$, $g(x)$, and $f(x) \cdot g(x)$. How are they related? ← You don't actually have to compute $f(x) \cdot g(x)$.

   (b) Same question, but in $\mathbb{Z}_7[x]$.

   (c) Same question, but in $\mathbb{Z}_9[x]$.

   Ans to (c): $\deg f(x) \cdot g(x) = 16$.

   (d) What's going on here? Can you *justify* it?

4. Consider the following theorem:

   Theorem. Let $f(x), g(x) \in R[x]$ where R is an integral domain, with $f(x), g(x) \neq 0$. Then $\deg f(x) \cdot g(x) = \deg f(x) + \deg g(x)$.

   Recall: An *integral domain* doesn't contain zero divisors.

   (a) Give an example that illustrates the theorem. **Hint:** See Problem #3.

   (b) Explain why the theorem is true. In particular, why must R be an integral domain?

5. (a) Find all units in $\mathbb{Z}_7[x]$, i.e., $f(x), g(x) \in \mathbb{Z}_7[x]$ such that $f(x) \cdot g(x) = 1$. **Ans:** 1, 2, 3, 4, 5, 6.

   (b) Find all units in $\mathbb{Z}[x]$; in $\mathbb{R}[x]$.

   (c) **Prove:** If R is an integral domain, then the only units in $R[x]$ are the units of R. ← Thus, non-constant polynomials are *not* units.

   Hint: Suppose $f(x) \cdot g(x) = 1$. Then $\deg f(x)$ and $\deg g(x)$ must be. . .

6. (a) Find all zero divisors in $\mathbb{Z}_7[x]$; in $\mathbb{Z}[x]$; in $\mathbb{R}[x]$. ← Are there any?

   (b) **Prove:** If R is an integral domain, then $R[x]$ is an integral domain.

   Hint: Let $f(x), g(x) \in R[x]$ be nonzero. Why must $f(x) \cdot g(x)$ also be nonzero?

7. Our friends are discussing the degree of the constant 0.

   Anita: "Why can't we say $\deg(0) = 0$? The zero polynomial is a constant, right?"

   Elizabeth: "But the theorem from Problem #4 fails if $\deg(0) = 0$."

   Hint: We have $x^2 \cdot 0 = 0$. Take the degree of both sides.

   What might Elizabeth mean?

8. (a) In $\mathbb{Z}_9[x]$, find $f(x)$ such that $(1 + 3x) \cdot f(x) = 1$.

   (b) In $\mathbb{Z}_6[x]$, find a nonzero $f(x)$ such that $(2 + 4x) \cdot f(x) = 0$.

   (c) How are $\mathbb{Z}_9[x]$ and $\mathbb{Z}_6[x]$ different from $\mathbb{Z}_7[x]$, $\mathbb{Z}[x]$, and $\mathbb{R}[x]$? Explain.

9. **(Some Food for Thought)**

   (a) Find *all* units in $\mathbb{Z}_9[x]$.

   (b) Find *all* zero divisors in $\mathbb{Z}_6[x]$.

10. **(More Food for Thought)**

    (a) In $\mathbb{Z}_9[x]$, find $f(x)$ such that $(1 - 3x) \cdot f(x) = 1$.

    (b) Same question, but in $\mathbb{Z}_{27}[x]$; in $\mathbb{Z}_{81}[x]$; in $\mathbb{Z}_{3^n}[x]$.