Abstract Algebra
Day 27 Class Work

1. We just saw that $\mathbb{Z}$ is an integral domain, while $\mathbb{Z}_{12}$ is not.
   (a) Find other examples of an integral domain.
   (b) Find some non-examples, i.e., commutative rings that are *not* an integral domain.

2. Let α and β be elements of an integral domain. If $\alpha \cdot \beta = 0$, what conclusion can you make about α and β? Explain your reasoning.

   Ans: $\alpha = 0$ or $\beta = 0$ (or possibly both).

3. **(Review of Group Theory)** In the *multiplicative* group U_{17}:
   (a) Find the multiplicative inverse of 6. **Ans:** $6^{-1} = 3$.
   (b) Use your answer in part (a) to solve the equation $6x = 10 \pmod{17}$. **Ans:** $x = 13$.

4. (a) In a multiplicative group G, explain why $ab = ac \implies b = c$. ← i.e., re-prove left cancellation in a group.
   (b) In the ring $\mathbb{Z}$, explain why

$$ab = ac, \ a \neq 0 \implies b = c. \qquad (\clubsuit)$$

   Remark: You can't use the same proof as in part (a), since most $a \in \mathbb{Z}$ do not have a multiplicative inverse. Instead, use the fact that $\mathbb{Z}$ is an integral domain.

   Hint: If $ab = ac$, then ??? $= 0$.

   (c) Use a counter-example to show how ($\clubsuit$) is false in $\mathbb{Z}_{12}$. How does your proof from part (b) fail when working in $\mathbb{Z}_{12}$?

5. (a) Find all the units in $\mathbb{Z}_7$. Do the same in $\mathbb{R}$, the set of real numbers.
   (b) Elizabeth says, "$\mathbb{Z}_7$ and $\mathbb{R}$ are *almost* multiplicative groups." What might she mean?

Definition: A commutative ring is called a *field* if every nonzero element is a unit.

Key: In a field, we can always "divide" (i.e., multiply by a^{-1}), except when $a = 0$.

Recall: A *unit* is a ring element that has a multiplicative inverse.

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6. In Problem #5, we saw that $\mathbb{Z}_7$ and $\mathbb{R}$ are fields.
   (a) Come up with a few other examples of a field.
   (b) Come up with some non-examples, i.e., commutative rings that are *not* a field.
   (c) For each field in part (a), determine if it's an integral domain. Any conjecture?
   (d) Find an integral domain that's not a field. ← Can you find more?

7. **Prove:** Every field is an integral domain.

8. Consider the ring $R = \mathbb{Z}_5[i] = \{a + bi \mid a, b \in \mathbb{Z}_5\}$. (Here, $i = \sqrt{-1}$ so that $i^2 = -1$.)
   (a) How many elements are in R? Explain your reasoning. **Ans:** 25 elements.
   (b) The element $1 + 4i \in R$ is a unit. Find its multiplicative inverse.
   (c) The element $\alpha = 1 - 2i \in R$ is a zero divisor. Find a nonzero $\beta \in R$ where $\alpha \cdot \beta = 0$.
   (d) Is R an integral domain, a field, or neither?

9. A ring element $a \in R$ is called a *self inverse under multiplication* if $a^2 = 1$.
   (a) Find all self inverses in $\mathbb{Z}$. **Ans:** 1 and -1.
   (b) Find all self inverses in $\mathbb{Z}_7$.
   (c) Find all self inverses in $\mathbb{Z}_{12}$. (**Hint:** It's not just 1 and $-1 = 11$.)
   (d) Find all self inverses in $\mathbb{Z}_{16}$.
   (e) Find all self inverses in $\mathbb{Z}_{13}$.
   (f) Any conjectures?

10. Determine all elements of an integral domain that are self inverses under multiplication. Explain your reasoning.

11. **(Some Food for Thought)** Consider the polynomial $f(x) = x^2 - 6x + 8$.
    (a) Factor the polynomial and use it to solve $x^2 - 6x + 8 = 0$ in $\mathbb{Z}$. **Ans:** $x = 2$ or 4.
    (b) Verify your result in part (a) by substituting each solution into $f(x)$.
    (c) Find all solutions to $x^2 - 6x + 8 = 0$ in $\mathbb{Z}_{15}$. (**Careful:** There are more than two.)
    (d) Anita says, "We found more than two solutions, since $\mathbb{Z}_{15}$ isn't an integral domain." What might she mean?
    (e) **(Challenge)** Find all solutions to $x^2 - 6x + 8 = 0$ in $\mathbb{Z}_{105}$. ← Yikes!