Abstract Algebra
Day 26 Class Work Solutions

Properties of a ring R:

1. R is closed under addition.
2. $(a+b)+c = a+(b+c)$ for all $a, b, c \in R$.
3. There exists $0 \in R$ such that $0+a=a$ and $a+0=a$ for all $a \in R$.
4. For $a \in R$, there exists $-a \in R$ s.t. $a+(-a)=0$ and $(-a)+a=0$.
5. $a+b=b+a$ for all $a, b \in R$.

← Properties 1 – 5 say that R is a commutative group under addition.

6. R is closed under multiplication.
7. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$.
8. There exists $1 \in R$ such that $1 \cdot a = a$ and $a \cdot 1 = a$ for all $a \in R$.
9. $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ and $(b+c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in R$.

← But R need *not* be commutative under mult.

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1. Other than $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{Z}_{12}$, come up with a few more examples of rings.

   Solution. Answers will vary. Some examples include: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_n$.

2. Recall that $M(\mathbb{R})$ is the set of 2×2 matrices with entries in $\mathbb{R}$.

   (a) Verify (briefly) that $M(\mathbb{R})$ satisfies the ring properties.

   Solution. With $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right]$, where all the matrix entries are in $\mathbb{R}$, we have $\alpha+\beta = \left[\begin{smallmatrix} a+w & b+x \\ c+y & d+z \end{smallmatrix}\right]$, where $a+w$, $b+x$, $c+y$, $d+z \in \mathbb{R}$, because $\mathbb{R}$ is closed under addition. Thus, $\alpha+\beta \in M(\mathbb{R})$, which shows the closure of $M(\mathbb{R})$ under addition.

   In $M(\mathbb{R})$, the additive and multiplicative identities are $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, respectively. The element α, as defined above, has an additive inverse $-\alpha = \left[\begin{smallmatrix} -a & -b \\ -c & -d \end{smallmatrix}\right] \in M(\mathbb{R})$. I'll leave the verification of the rest of the properties up to you.

   (b) Find an example to show that $M(\mathbb{R})$ is *not* commutative under multiplication.

   Note: Thus, $M(\mathbb{R})$ is an example of a *non-commutative* ring.

   Solution. Answers will vary. For example, with $\alpha = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$, we have $\alpha\beta = \left[\begin{smallmatrix} 19 & 22 \\ 43 & 50 \end{smallmatrix}\right]$ and $\beta\alpha = \left[\begin{smallmatrix} 23 & 34 \\ 31 & 46 \end{smallmatrix}\right]$. Therefore, $\alpha\beta \neq \beta\alpha$.

3. Use the table below to record your answers to parts (a) and (b).

   (a) Classify each *nonzero* element of $\mathbb{Z}_{12}$ as a unit, a zero divisor, neither, or both.

   ← 0 is always neither.

   (b) Do the same in $\mathbb{Z}_7$; and in $\mathbb{Z}$; and in $\mathbb{R}$.

   Solution.

	$\mathbb{Z}_{12}$	$\mathbb{Z}_7$	$\mathbb{Z}$	$\mathbb{R}$
units	1, 5, 7, 11	1, 2, 3, 4, 5, 6	1, −1	All nonzero elements
ZDs	2, 3, 4, 6, 8, 9, 10	none	none	none
neither	none	none	all except ±1	none
both	none	none	none	none

(c) In $M(\mathbb{R})$, find examples (if possible) of a unit, a zero divisor, neither, or both.

← Or explain why such an element doesn't exist.

Solution.

- Let $\alpha = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 1 & -2 \\ 0 & 1 \end{smallmatrix}\right]$. Then $\alpha \cdot \beta = \varepsilon$, where $\varepsilon = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is the multiplicative identity. Thus, α and β are both units in $M(\mathbb{R})$.
- Let $\gamma = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\delta = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then $\gamma \cdot \delta = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ (i.e., the additive identity). Thus, γ and δ are zero divisors.
- There is no nonzero element that's neither. Can you explain why?
- There is no element that's both. See Problem #8.

(d) What conjectures do you have? Can you prove them?

4. Explain why each R is *not* a ring.

(a) $R = \{a \in \mathbb{R} \mid a > 0\}$, i.e., the set of positive real numbers.

Solution. The positive real numbers do not have additive inverses in R.

(b) $R = 5\mathbb{Z}$ with integer addition and multiplication.

Solution. The multiplicative identity 1 is not in R.

(c) $R = U_{13}$ with addition and multiplication modulo 13.

Solution. R is not closed under addition (e.g., $9 + 4 = 0 \notin R$); 0 is not in R.

5. (a) Quick, what's $a \cdot 0$ in any ring?

(b) Using only the ring properties, prove: In a ring R, $a \cdot 0 = 0$ for all $a \in R$.

Hint: Start with $0 + 0 = 0$.

Solution. See Theorem 26.9 and its proof in the textbook.

6. Consider the ring $\mathbb{Z}_3[i] = \{a + bi \mid a, b \in \mathbb{Z}_3\}$, where $i = \sqrt{-1}$ so that $i^2 = -1$. Here are some examples that illustrate how to add and multiply in this ring:

- $(1 + 2i) + (2 + i) = 3 + 3i = 0 + 0i$ (or just 0), since $3 = 0$ in $\mathbb{Z}_3$.
- $(1 + 2i) \cdot (2 + i) = 1 \cdot 2 + 1 \cdot i + 2i \cdot 2 + 2i \cdot i$

$$= 2 + 5i + 2i^2 = 2 + 5i + 2(-1) = 0 + 5i = 2i, \text{ since } 5 = 2 \text{ in } \mathbb{Z}_3.$$

(a) Find all elements of $\mathbb{Z}_3[i]$. How many of them are there?

Ans: 9 elements.

Solution. 9 elements: 0, 1, 2, i, $1 + i$, $2 + i$, $2i$, $1 + 2i$, $2 + 2i$.

(b) $1 + 2i \in \mathbb{Z}_3[i]$ has a multiplicative inverse. Find it.

Solution. We must find $a + bi \in \mathbb{Z}_3[i]$ such that $(1 + 2i) \cdot (a + bi) = 1$. We have

$$(1 + 2i) \cdot (a + bi) = (a - 2b) + (2a + b)i.$$

Setting this equal to 1 (or $1 + 0i$) implies $a - 2b = 1$ and $2a + b = 0$. Solving this system of equations in $\mathbb{Z}_3$, we obtain $a = 2$ and $b = 2$. Thus, $(1 + 2i)^{-1} = 2 + 2i$.

(c) Classify each *nonzero* element of $\mathbb{Z}_3[i]$ as a unit or a zero divisor.

Solution. Every nonzero element is a unit, as shown below:

$$1 \cdot 1 = 1,\ 2 \cdot 2 = 1,\ i \cdot (-i) = 1,\ (1 + 2i) \cdot (2 + 2i) = 1,\ (2 + i) \cdot (1 + i) = 1.$$

7. (a) Find all zero divisors in $\mathbb{Z}_{12}$. Explain your reasoning.

Solution. We have $2 \cdot 6 = 0$, $3 \cdot 4 = 0$, $8 \cdot 9 = 0$, $10 \cdot 6 = 0$. Thus 2, 3, 4, 6, 8, 9, 10 are zero divisors in $\mathbb{Z}_{12}$. The other non-zero elements of $\mathbb{Z}_{12}$ (i.e., 1, 5, 7, 11) are all units, and thus cannot be zero divisors by Problem #8.

(b) Repeat part (a) with $\mathbb{Z}_{15}$.

Solution. We have $3 \cdot 5 = 0$, $6 \cdot 10 = 0$, $9 \cdot 5 = 0$, $12 \cdot 5 = 0$. Thus 3, 5, 6, 9, 10, 12 are zero divisors in $\mathbb{Z}_{15}$. The other non-zero elements of $\mathbb{Z}_{15}$ (i.e., 1, 2, 4, 7, 8, 11, 13, 14) are all units, and thus cannot be zero divisors by Problem #8.

(c) Repeat part (a) with $\mathbb{Z}_{17}$.

Solution. There are no zero divisors in $\mathbb{Z}_{17}$.

(d) What conjectures do you have? Can you prove them?

8. **Prove:** Let α be a ring element. Then α cannot be both a unit and a zero divisor.

Solution. See Theorem 26.18 and its proof in the textbook.