Abstract Algebra
Day 26 Class Work

Properties of a ring R:

1. R is closed under addition.
2. $(a+b)+c = a+(b+c)$ for all a, b, $c \in R$.
3. There exists $0 \in R$ such that $0 + a = a$ and $a + 0 = a$ for all $a \in R$.
4. For $a \in R$, there exists $-a \in R$ s.t. $a + (-a) = 0$ and $(-a) + a = 0$.
5. $a + b = b + a$ for all a, $b \in R$.

← Properties 1 – 5 say that R is a commutative group under addition.

6. R is closed under multiplication.
7. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, $c \in R$.
8. There exists $1 \in R$ such that $1 \cdot a = a$ and $a \cdot 1 = a$ for all $a \in R$.
9. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all a, b, $c \in R$.

← But R need *not* be commutative under mult.

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1. Other than $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{Z}_{12}$, come up with a few more examples of rings.
2. Recall that $M(\mathbb{R})$ is the set of 2×2 matrices with entries in $\mathbb{R}$.
   (a) Verify (briefly) that $M(\mathbb{R})$ satisfies the ring properties.
   (b) Find an example to show that $M(\mathbb{R})$ is *not* commutative under multiplication.
   Note: Thus, $M(\mathbb{R})$ is an example of a *non-commutative* ring.
3. Use the table below to record your answers to parts (a) and (b).
   (a) Classify each *nonzero* element of $\mathbb{Z}_{12}$ as a unit, a zero divisor, neither, or both. ← 0 is always neither.
   (b) Do the same in $\mathbb{Z}_7$; and in $\mathbb{Z}$; and in $\mathbb{R}$.

	$\mathbb{Z}_{12}$	$\mathbb{Z}_7$	$\mathbb{Z}$	$\mathbb{R}$
units				
ZDs				
neither				
both				

   (c) In $M(\mathbb{R})$, find examples (if possible) of a unit, a zero divisor, neither, or both. ← Or explain why such an element doesn't exist.
   (d) What conjectures do you have? Can you prove them?
4. Explain why each R is *not* a ring.
   (a) $R = \{a \in \mathbb{R} \mid a > 0\}$, i.e., the set of positive real numbers.
   (b) $R = 5\mathbb{Z}$ with integer addition and multiplication.
   (c) $R = U_{13}$ with addition and multiplication modulo 13.

5. (a) Quick, what's $a \cdot 0$ in any ring?

   (b) Using only the ring properties, prove: In a ring R, $a \cdot 0 = 0$ for all $a \in R$. **Hint:** Start with $0 + 0 = 0$.

6. Consider the ring $\mathbb{Z}_3[i] = \{a + bi \mid a, b \in \mathbb{Z}_3\}$, where $i = \sqrt{-1}$ so that $i^2 = -1$. Here are some examples that illustrate how to add and multiply in this ring:

   - $(1 + 2i) + (2 + i) = 3 + 3i = 0 + 0i$ (or just 0), since $3 = 0$ in $\mathbb{Z}_3$.
   - $$\begin{aligned}(1 + 2i) \cdot (2 + i) &= 1 \cdot 2 + 1 \cdot i + 2i \cdot 2 + 2i \cdot i \\ &= 2 + 5i + 2i^2 = 2 + 5i + 2(-1) = 0 + 5i = 2i, \text{ since } 5 = 2 \text{ in } \mathbb{Z}_3.\end{aligned}$$

   (a) Find all elements of $\mathbb{Z}_3[i]$. How many of them are there? **Ans:** 9 elements.

   (b) $1 + 2i \in \mathbb{Z}_3[i]$ has a multiplicative inverse. Find it.

   (c) Classify each *nonzero* element of $\mathbb{Z}_3[i]$ as a unit or a zero divisor.

7. (a) Find all zero divisors in $\mathbb{Z}_{12}$. Explain your reasoning.

   (b) Repeat part (a) with $\mathbb{Z}_{15}$.

   (c) Repeat part (a) with $\mathbb{Z}_{17}$.

   (d) What conjectures do you have? Can you prove them?

8. **Prove:** Let α be a ring element. Then α cannot be both a unit and a zero divisor.