Abstract Algebra
Day 25 Class Work Solutions

1. Consider $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$. Let $K = \ker \lambda = \{1, 3, 9\}$. ← λ is a homomorphism.

   (a) Find all distinct cosets of K in U_{13}.

   Solution.
   - $1K = 3K = 9K = \{1, 3, 9\}$ (original subgroup)
   - $2K = 5K = 6K = \{2, 5, 6\}$
   - $4K = 10K = 12K = \{4, 10, 12\}$
   - $7K = 8K = 11K = \{7, 8, 11\}$

   (b) How do your cosets compare with how λ partitions U_{13}? Are you surprised by this? ← I'm (still) surprised!

   Solution. The cosets of $K = \ker \lambda$ partition the domain in the same way that the homomorphism λ does.

2. Our friends just finished working on Problem #1. Here's their recap:

   Elizabeth: So it looks like the cosets of $K = \ker \lambda$ partition the domain in the same way that the homomorphism λ does.

   Anita: Yeah, which is why the subsets created by λ are *equal-sized.*

   What might they mean? Explain their statements.

   Solution. All the cosets of K have the same size, namely $\#K$. Thus, as Anita noted, the subsets created by λ are equal-sized.

3. Consider again $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$. Note that $\lambda(2) = 2^3 = 8$.

   (a) Using the cosets from Problem #1, find all other $a \in U_{13}$ such that $\lambda(a) = 8$. **Ans:** $a = 5$ and 6.

   Solution. We have $2K = \{2, 5, 6\}$. Thus, we should have $\lambda(2) = \lambda(5) = \lambda(6)$.

   (b) Verify your answer in part (a) by computing $\lambda(a) = a^3$ for each $a \in U_{13}$ you found.

   Solution. Indeed, we have $\lambda(5) = 5^3 = 8$ and $\lambda(6) = 6^3 = 8$.

4. Suppose $\varphi : U_{17} \to U_{17}$ is a homomorphism with kernel $K = \{1, 4, 13, 16\}$.

   (a) **(Optional)** Verify that K is indeed a subgroup of the domain U_{17}.

   (b) Find all distinct cosets of K in U_{17}.

   Solution.
   - $1K = 4K = 13K = 16K = \{1, 4, 13, 16\}$ (original subgroup)
   - $2K = 8K = 9K = 15K = \{2, 8, 9, 15\}$
   - $3K = 12K = 5K = 14K = \{3, 12, 5, 14\}$
   - $6K = 7K = 10K = 11K = \{6, 7, 10, 11\}$

   (c) Suppose $\varphi(10) = 4$. Find all other $a \in U_{17}$ such that $\varphi(a) = 4$. How do you know that you've found *all* such elements? **Ans:** $a = 6$, 11, and 7.

   Solution. Since $10K = \{6, 7, 10, 11\}$, we have $\varphi(6) = \varphi(7) = \varphi(10) = \varphi(11)$.

5. Let $\theta : G \to H$ be a group homomorphism with $K = \ker \theta$. Let $g \in G$ such that $\theta(g) = h$ where $h \in H$. Given $a \in G$, prove each of the following.

   (a) If $a \in gK$, then $\theta(a) = h$ (i.e., every element of coset gK maps to h).

Hint: An element $k \in K$ satisfies $\theta(k) = \varepsilon_H$.

Proof. Assume $a \in gK$ so that $a = gk$ for some $k \in K$. Note that $\theta(k) = \varepsilon_H$, since k is in the kernel of θ. Thus, $\theta(a) = \theta(gk) = \theta(g)\theta(k) = h\varepsilon_H = h$, as desired. ■

(b) **(Optional)** If $a \notin gK$, then $\theta(a) \neq h$ (i.e., *only* the elements of gK map to h).

← You might want to prove the contrapositive here.

Proof. We will prove the contrapositive: If $\theta(a) = h$, then $a \in gK$. Assume $\theta(a) = h$. Then $\theta(a) = \theta(g)$ so that $\theta(g^{-1}a) = \theta(g)^{-1}\theta(a) = \theta(g)^{-1}\theta(g) = \varepsilon_H$. Therefore, $g^{-1}a \in K$ so that $aK = gK$. Since $a \in aK$, we have $a \in gK$ as desired. ■

(c) Elizabeth says, "In parts (a) and (b), we proved what I noticed in Problem #2." What might she mean?

Solution. Parts (a) and (b) show that the cosets of K partition the domain G in the same way that θ does.

6. Consider yet again $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$ with kernel K. We've seen that it has image $\operatorname{im}\lambda = \{1, 8, 12, 5\}$. And in Problem #1, you found that the distinct cosets of K are $1K$, $2K$, $4K$, and $7K$.

(a) Create the group tables for U_{13}/K and $\operatorname{im}\lambda$.

Solution.

$\cdot$	$1K$	$2K$	$4K$	$7K$
$1K$	$1K$	$2K$	$4K$	$7K$
$2K$	$2K$	$4K$	$7K$	$1K$
$4K$	$4K$	$7K$	$1K$	$2k$
$7K$	$7K$	$1K$	$2K$	$4K$

$\cdot$	1	8	12	5
1	1	8	12	5
8	8	12	5	1
12	12	5	1	8
5	5	1	8	12

(b) Verify that the two tables in part (a) are essentially the same. How are the groups U_{13}/K and $\operatorname{im}\lambda$ related?

Ans: They're isomorphic.

Solution. The two tables are essentially the same, so that the two groups are isomorphic, i.e., $U_{13}/K \cong \operatorname{im}\lambda$, where $gK \in U_{13}/K$ corresponds to $\lambda(g) \in \operatorname{im}\lambda$.

7. Consider again the homomorphism $\varphi : U_{17} \to U_{17}$ with kernel $K = \{1, 4, 13, 16\}$. And as in Problem #4, suppose $\varphi(10) = 4$.

(a) Find the value of $\varphi(a)$ for all $a \in U_{17}$.

Hint: φ is operation preserving, i.e., $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$, for all $a, b \in U_{17}$.

Solution.

- Since $\ker\varphi = \{1, 4, 13, 16\}$, we have $\varphi(1) = \varphi(4) = \varphi(13) = \varphi(16) = 1$.
- We saw in Problem #4(c) that $\varphi(6) = \varphi(7) = \varphi(10) = \varphi(11) = 4$.
- We have $10^2 = 15 \pmod{17}$. Thus, $\varphi(15) = \varphi(10^2) = \varphi(10)^2 = 4^2 = 16$. And since $15K = \{2, 8, 9, 15\}$, we have $\varphi(2) = \varphi(8) = \varphi(9) = \varphi(15) = 16$.
- We have $10^3 = 14 \pmod{17}$. Thus, $\varphi(14) = \varphi(10^3) = \varphi(10)^3 = 4^3 = 13$. And since $14K = \{3, 12, 5, 14\}$, we have $\varphi(3) = \varphi(12) = \varphi(5) = \varphi(14) = 13$.

(b) Find the image $\operatorname{im}\varphi$.

Ans for (b): $\operatorname{im}\varphi = \{1, 16, 13, 4\}$.

Solution. We have $\operatorname{im}\varphi = \{1, 16, 13, 4\}$.

(c) Create the group tables for U_{17}/K and $\operatorname{im}\varphi$.

Solution.

$\cdot$	$1K$	$2K$	$3K$	$6K$
$1K$	$1K$	$2K$	$3K$	$6K$
$2K$	$2K$	$1K$	$6K$	$3K$
$3K$	$3K$	$6K$	$2K$	$1K$
$6K$	$6K$	$3K$	$1K$	$2K$

$\cdot$	1	16	13	4
1	1	16	13	4
16	16	1	4	13
13	13	4	16	1
4	4	13	1	16

(d) Verify that the two tables in part (c) are essentially the same. How are the groups U_{17}/K and $\operatorname{im}\varphi$ related?

Solution. The two tables are essentially the same, so that the two groups are isomorphic, i.e., $U_{17}/K \cong \operatorname{im}\varphi$, where $gK \in U_{17}/K$ corresponds to $\varphi(g) \in \operatorname{im}\varphi$.

(e) **(Optional)** Find a *formula* for the function φ and verify that it's a homomorphism.

8. Let $\gamma : U_{31} \to U_{31}$ be a homomorphism with kernel $K = \{1, 5, 6, 25, 26, 30\}$.

(a) Find all distinct cosets of K in U_{31}.

(b) Suppose $\gamma(10) = 2$. Find all other $a \in U_{31}$ such that $\gamma(a) = 2$.

(c) Find the value of $\gamma(a)$ for all $a \in U_{31}$.

(d) Find the image $\operatorname{im}\gamma$.

Ans for (d): $\operatorname{im}\gamma = \{1, 2, 4, 8, 16\}$.

(e) Create and compare the group tables for U_{31}/K and $\operatorname{im}\gamma$.

9. Consider $\delta : G(\mathbb{Z}_{10}) \to U_{10}$ where $\delta(\alpha) = \det\alpha$ for all $\alpha \in G(\mathbb{Z}_{10})$.

Recall: $G(\mathbb{Z}_{10})$ is the multiplicative group of invertible 2×2 matrices with entries in $\mathbb{Z}_{10}$.

(a) List a few elements of the kernel K.

(b) How does δ partition the domain $G(\mathbb{Z}_{10})$?

(c) Describe all distinct cosets of K.

(d) Complete the group tables for $G(\mathbb{Z}_{10})/K$ and $\operatorname{im}\delta$. How do the two tables compare?

Solution. See Section 25.2 in the textbook for details.