# Abstract Algebra
## Day 25 Class Work

1. Consider $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$. Let $K = \ker \lambda = \{1, 3, 9\}$. ← $\lambda$ is a homomorphism.
   (a) Find all distinct cosets of $K$ in $U_{13}$.
   (b) How do your cosets compare with how $\lambda$ partitions $U_{13}$? Are you surprised by this? ← I'm (still) surprised!

2. Our friends just finished working on Problem #1. Here's their recap:

   **Elizabeth:** So it looks like the cosets of $K = \ker \lambda$ partition the domain in the same way that the homomorphism $\lambda$ does.

   **Anita:** Yeah, which is why the subsets created by $\lambda$ are *equal-sized*.

   What might they mean? Explain their statements.

3. Consider again $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$. Note that $\lambda(2) = 2^3 = 8$.
   (a) Using the cosets from Problem #1, find all other $a \in U_{13}$ such that $\lambda(a) = 8$. **Ans:** $a = 5$ and 6.
   (b) Verify your answer in part (a) by computing $\lambda(a) = a^3$ for each $a \in U_{13}$ you found.

4. Suppose $\varphi : U_{17} \to U_{17}$ is a homomorphism with kernel $K = \{1, 4, 13, 16\}$.
   (a) **(Optional)** Verify that $K$ is indeed a subgroup of the domain $U_{17}$.
   (b) Find all distinct cosets of $K$ in $U_{17}$.
   (c) Suppose $\varphi(10) = 4$. Find all other $a \in U_{17}$ such that $\varphi(a) = 4$. How do you know that you've found *all* such elements? **Ans:** $a = 6$, 11, and 7.

5. Let $\theta : G \to H$ be a group homomorphism with $K = \ker \theta$. Let $g \in G$ such that $\theta(g) = h$ where $h \in H$. Given $a \in G$, prove each of the following.
   (a) If $a \in gK$, then $\theta(a) = h$ (i.e., every element of coset $gK$ maps to $h$).

   **Hint:** An element $k \in K$ satisfies $\theta(k) = \varepsilon_H$.

   (b) **(Optional)** If $a \notin gK$, then $\theta(a) \neq h$ (i.e., *only* the elements of $gK$ map to $h$). ← You might want to prove the contrapositive here.
   (c) Elizabeth says, "In parts (a) and (b), we proved what I noticed in Problem #2." What might she mean?

6. Consider yet again $\lambda : U_{13} \to U_{13}$ where $\lambda(a) = a^3$ for all $a \in U_{13}$ with kernel $K$. We've seen that it has image $\operatorname{im} \lambda = \{1, 8, 12, 5\}$. And in Problem #1, you found that the distinct cosets of $K$ are $1K$, $2K$, $4K$, and $7K$.
   (a) Create the group tables for $U_{13}/K$ and $\operatorname{im} \lambda$.

| $\cdot$ | $1K$ | $2K$ | $4K$ | $7K$ |
|---|---|---|---|---|
| $1K$ | | | | |
| $2K$ | | | | |
| $4K$ | | | | |
| $7K$ | | | | |

| $\cdot$ | 1 | 8 | 12 | 5 |
|---|---|---|---|---|
| 1 | | | | |
| 8 | | | | |
| 12 | | | | |
| 5 | | | | |

   (b) Verify that the two tables in part (a) are essentially the same. How are the groups $U_{13}/K$ and $\operatorname{im} \lambda$ related? **Ans:** They're isomorphic.

7. Consider again the homomorphism $\varphi : U_{17} \to U_{17}$ with kernel $K = \{1,\ 4,\ 13,\ 16\}$. And as in Problem #4, suppose $\varphi(10) = 4$.

   (a) Find the value of $\varphi(a)$ for all $a \in U_{17}$.

   **Hint:** $\varphi$ is operation preserving, i.e., $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$, for all $a,\ b \in U_{17}$.

   (b) Find the image $\operatorname{im}\varphi$.

   **Ans for (b):** $\operatorname{im}\varphi = \{1,\ 16,\ 13,\ 4\}$.

   (c) Create the group tables for $U_{17}/K$ and $\operatorname{im}\varphi$.

| $\cdot$ | $1K$ | $2K$ | $3K$ | $6K$ |
|---|---|---|---|---|
| $1K$ | | | | |
| $2K$ | | | | |
| $3K$ | | | | |
| $6K$ | | | | |

| $\cdot$ | 1 | 16 | 13 | 4 |
|---|---|---|---|---|
| 1 | | | | |
| 16 | | | | |
| 13 | | | | |
| 4 | | | | |

   (d) Verify that the two tables in part (c) are essentially the same. How are the groups $U_{17}/K$ and $\operatorname{im}\varphi$ related?

   (e) **(Optional)** Find a *formula* for the function $\varphi$ and verify that it's a homomorphism.

8. Let $\gamma : U_{31} \to U_{31}$ be a homomorphism with kernel $K = \{1,\ 5,\ 6,\ 25,\ 26,\ 30\}$.

   (a) Find all distinct cosets of $K$ in $U_{31}$.

   (b) Suppose $\gamma(10) = 2$. Find all other $a \in U_{31}$ such that $\gamma(a) = 2$.

   (c) Find the value of $\gamma(a)$ for all $a \in U_{31}$.

   (d) Find the image $\operatorname{im}\gamma$.

   **Ans for (d):** $\operatorname{im}\gamma = \{1, 2, 4, 8, 16\}$.

   (e) Create and compare the group tables for $U_{31}/K$ and $\operatorname{im}\gamma$.

9. Consider $\delta : G(\mathbb{Z}_{10}) \to U_{10}$ where $\delta(\alpha) = \det\alpha$ for all $\alpha \in G(\mathbb{Z}_{10})$.

   **Recall:** $G(\mathbb{Z}_{10})$ is the multiplicative group of invertible $2 \times 2$ matrices with entries in $\mathbb{Z}_{10}$.

   (a) List a few elements of the kernel $K$.

   (b) How does $\delta$ partition the domain $G(\mathbb{Z}_{10})$?

   (c) Describe all distinct cosets of $K$.

   (d) Complete the group tables for $G(\mathbb{Z}_{10})/K$ and $\operatorname{im}\delta$. How do the two tables compare?