4K

7K

## Abstract Algebra Day 25 Class Work

- 1. Consider  $\lambda : U_{13} \to U_{13}$  where  $\lambda(a) = a^3$  for all  $a \in U_{13}$ . Let  $K = \ker \lambda = \{1, 3, 9\}$ .  $\leftarrow \lambda$  is a homomorphism. (a) Find all distinct cosets of K in  $U_{13}$ . (b) How do your cosets compare with how  $\lambda$  partitions  $U_{13}$ ? Are you surprised by this?  $\leftarrow$  I'm (still) surprised! 2. Our friends just finished working on Problem #1. Here's their recap: **Elizabeth:** So it looks like the cosets of  $K = \ker \lambda$  partition the domain in the same way that the homomorphism  $\lambda$  does. Anita: Yeah, which is why the subsets created by  $\lambda$  are equal-sized. What might they mean? Explain their statements. 3. Consider again  $\lambda: U_{13} \to U_{13}$  where  $\lambda(a) = a^3$  for all  $a \in U_{13}$ . Note that  $\lambda(2) = 2^3 = 8$ . (a) Using the cosets from Problem #1, find all other  $a \in U_{13}$  such that  $\lambda(a) = 8$ . Ans: a = 5 and 6(b) Verify your answer in part (a) by computing  $\lambda(a) = a^3$  for each  $a \in U_{13}$  you found. 4. Suppose  $\varphi: U_{17} \to U_{17}$  is a homomorphism with kernel  $K = \{1, 4, 13, 16\}$ . (a) (**Optional**) Verify that K is indeed a subgroup of the domain  $U_{17}$ . (b) Find all distinct cosets of K in  $U_{17}$ . (c) Suppose  $\varphi(10) = 4$ . Find all other  $a \in U_{17}$  such that  $\varphi(a) = 4$ . How do you know Ans: a = 6, 11, and 7. that you've found *all* such elements? 5. Let  $\theta: G \to H$  be a group homomorphism with  $K = \ker \theta$ . Let  $g \in G$  such that  $\theta(g) = h$ where  $h \in H$ . Given  $a \in G$ , prove each of the following. (a) If  $a \in qK$ , then  $\theta(a) = h$  (i.e., every element of coset qK maps to h). **Hint:** An element  $k \in K$  satisfies  $\theta(k) = \varepsilon_H$ . (b) (**Optional**) If  $a \notin gK$ , then  $\theta(a) \neq h$  (i.e., only the elements of gK map to h).  $\leftarrow$  You might want to prove the contrapositive here. (c) Elizabeth says, "In parts (a) and (b), we proved what I noticed in Problem #2." What might she mean? 6. Consider yet again  $\lambda: U_{13} \to U_{13}$  where  $\lambda(a) = a^3$  for all  $a \in U_{13}$  with kernel K. We've seen that it has image im  $\lambda = \{1, 8, 12, 5\}$ . And in Problem #1, you found that the distinct cosets of K are 1K, 2K, 4K, and 7K. (a) Create the group tables for  $U_{13}/K$  and im  $\lambda$ . 1K2K4K127K1 8 5 1K1 8 2K
  - (b) Verify that the two tables in part (a) are essentially the same. How are the groups Ans: They're isomorphic.  $U_{13}/K$  and im  $\lambda$  related?

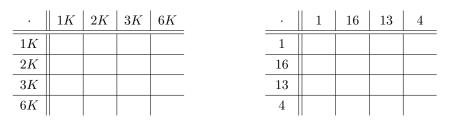
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- 7. Consider again the homomorphism  $\varphi: U_{17} \to U_{17}$  with kernel  $K = \{1, 4, 13, 16\}$ . And as in Problem #4, suppose  $\varphi(10) = 4$ .
  - (a) Find the value of  $\varphi(a)$  for all  $a \in U_{17}$ .

**Hint:**  $\varphi$  is operation preserving, i.e.,  $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$ , for all  $a, b \in U_{17}$ .

- (b) Find the image im  $\varphi$ .
- (c) Create the group tables for  $U_{17}/K$  and  $\operatorname{im} \varphi$ .



- (d) Verify that the two tables in part (c) are essentially the same. How are the groups  $U_{17}/K$  and im  $\varphi$  related?
- (e) (**Optional**) Find a *formula* for the function  $\varphi$  and verify that it's a homomorphism.
- 8. Let  $\gamma: U_{31} \to U_{31}$  be a homomorphism with kernel  $K = \{1, 5, 6, 25, 26, 30\}$ .
  - (a) Find all distinct cosets of K in  $U_{31}$ .
  - (b) Suppose  $\gamma(10) = 2$ . Find all other  $a \in U_{31}$  such that  $\gamma(a) = 2$ .
  - (c) Find the value of  $\gamma(a)$  for all  $a \in U_{31}$ .
  - (d) Find the image im  $\gamma$ .
  - (e) Create and compare the group tables for  $U_{31}/K$  and im  $\gamma$ .
- 9. Consider  $\delta : G(\mathbb{Z}_{10}) \to U_{10}$  where  $\delta(\alpha) = \det \alpha$  for all  $\alpha \in G(\mathbb{Z}_{10})$ .

**Recall:**  $G(\mathbb{Z}_{10})$  is the multiplicative group of invertible  $2 \times 2$  matrices with entries in  $\mathbb{Z}_{10}$ .

- (a) List a few elements of the kernel K.
- (b) How does  $\delta$  partition the domain  $G(\mathbb{Z}_{10})$ ?
- (c) Describe all distinct cosets of K.
- (d) Complete the group tables for  $G(\mathbb{Z}_{10})/K$  and  $\operatorname{im} \delta$ . How do the two tables compare?

im  $\varphi = \{1, 16, 13, 4\}.$ 

Ans for (b):

Ans for (d): im  $\gamma = \{1, 2, 4, 8, 16\}.$