Abstract Algebra
Day 24 Class Work Solutions

Here's the group table for D_4. Recall that for σ, $\tau \in D_4$, the "product" $\sigma \circ \tau$ is the entry in row σ and column τ. For example, the product $d \circ r_{90} = v$ is shown in bold.

← You'll be using this a lot.

$\circ$	ε	$\boldsymbol{r_{90}}$	r_{180}	r_{270}	h	v	d	d'
ε	ε	r_{90}	r_{180}	r_{270}	h	v	d	d'
r_{90}	r_{90}	r_{180}	r_{270}	ε	d'	d	h	v
r_{180}	r_{180}	r_{270}	ε	r_{90}	v	h	d'	d
r_{270}	r_{270}	ε	r_{90}	r_{180}	d	d'	v	h
h	h	d	v	d'	ε	r_{180}	r_{90}	r_{270}
v	v	d'	h	d	r_{180}	ε	r_{270}	r_{90}
$\boldsymbol{d}$	d	$\boldsymbol{v}$	d'	h	r_{270}	r_{90}	ε	r_{180}
d'	d'	h	d	v	r_{90}	r_{270}	r_{180}	ε

1. We'll analyze how the shortcut *fails*. Let $H = \{\varepsilon, v\}$ be a subgroup of D_4.

   (a) Compute the cosets $r_{270}H$ and dH.

   Ans: $r_{270}H = \{r_{270}, d'\}$.

   Solution. We have $r_{270}H = \{r_{270}, d'\}$ and $dH = \{d, r_{90}\}$.

   (b) Compute $r_{270}H \cdot dH$ by multiplying each element of $r_{270}H$ by those of dH.

   Solution. We have...

$$\begin{aligned} r_{270}H \cdot dH &= \{r_{270}, d'\} \cdot \{d, r_{90}\} \\ &= \{r_{270}d, r_{270}r_{90}, d'\,d, d'\,r_{90}\} \\ &= \{v, \varepsilon, r_{180}, h\} \end{aligned}$$

   (c) Anita says, "The product $r_{270}H \cdot dH$ is definitely *not* a coset of H, because it has too many elements." What might she mean?

   ← The CM shortcut fails!

   Solution. A coset of H must have the same number of elements as H, namely 2. However, the product $r_{270}H \cdot dH$ has 4 elements, and thus cannot be a coset of H.

   (d) Compute $(r_{270} \cdot d)H$ and verify that $(r_{270} \cdot d)H \subseteq r_{270}H \cdot dH$.

   Ans: $(r_{270} \cdot d)H = \{v, \varepsilon\}$.

   Note: In fact, this inclusion *always* holds. (See Problem #2.)

   Solution. We have $(r_{270} \cdot d)H = \{v, \varepsilon\}$ and $r_{270}H \cdot dH = \{v, \varepsilon, r_{180}, h\}$, so that $(r_{270} \cdot d)H$ is indeed a subset of $r_{270}H \cdot dH$.

2. **Prove:** Let G be a group and H a subgroup. For $a, b \in G$, define the coset product by

$$aH \cdot bH = \{\alpha \cdot \beta \mid \alpha \in aH,\ \beta \in bH\}.$$

   Then $(ab)H \subseteq aH \cdot bH$. (**Hint:** Let $x \in (ab)H$ and show that $x \in aH \cdot bH$.)

   Ans: $x = (ab)h$
   $= a\varepsilon \cdot bh$.

   PROOF. Let $x \in (ab)H$ so that $x = (ab)h$ for some $h \in H$. Then, $x = (ab)h = a\varepsilon \cdot bh$, where $a\varepsilon \in aH$ and $bh \in bH$. Thus, $x \in aH \cdot bH$ so that $(ab)H \subseteq aH \cdot bH$. ■

What we know so far:

Since $(ab)H \subseteq aH \cdot bH$ always holds, the key to coset multiplication is $aH \cdot bH \subseteq (ab)H$. When does *this* set inclusion hold? We'll explore in the next few questions!

---

3. Let G be a *commutative* group, H a subgroup, and $a,\, b \in G$.

   (a) Prove that $aH \cdot bH \subseteq (ab)H$.

   Hint: Let $\alpha\beta \in aH \cdot bH$, where $\alpha \in aH$ and $\beta \in bH$. Show that $\alpha\beta \in (ab)H$. ← So, α and β look like. . .

   PROOF. Suppose $\alpha\beta \in aH \cdot bH$, where $\alpha \in aH$ and $\beta \in bH$. Thus, $\alpha = ah$ and $\beta = bk$ for some $h,\, k \in H$. Since G is commutative, $hb = bh$ so that

   $$\alpha\beta = (ah)(bk) = a\boldsymbol{(hb)}k = a\boldsymbol{(bh)}k = (ab)(hk) \in (ab)H.$$

   Thus, $aH \cdot bH \subseteq (ab)H$. ■

   (b) Where in your proof in part (a) did you use the fact that G is commutative?

   Solution. We used $hb = bh$ to show that $\alpha\beta \in (ab)H$.

4. Let $K = \{\varepsilon,\, r_{90},\, r_{180},\, r_{270}\}$ be a subgroup of D_4.

   (a) For each $a \in D_4$, compute the left coset aK and the right coset Ka.

   Hint: What if a is a rotation? What if it's a reflection? ← You shouldn't have to do much calculation here.

   Solution. We have

   $$aK = Ka = \begin{cases} K & \text{if } a \in K \text{ (i.e., } a = \text{rotation)} \\ D_4 - K & \text{if } a \notin K \text{ (i.e., } a = \text{reflection)} \end{cases}$$

   Here, $D_4 - K$ is the set of elements in D_4 that are *not* in K. In other words, $D_4 - K = \{h,\, v,\, d,\, d'\}$, the subset of D_4 containing all reflections.

   (b) **True or False:** $aK = Ka$ for all $a \in D_4$.

   Solution. True. See the solution for part (a).

   (c) **True or False:** $aK = Ka$ means $ak = ka$ for each $k \in K$.

   Note: In other words, does coset equality imply element-by-element equality?

   Solution. False. For instance, we have $hK = Kh$ where

   $$hK = \{h\varepsilon,\, hr_{90},\, hr_{180},\, hr_{270}\} = \{h,\, \boldsymbol{d},\, v,\, d'\}$$
   $$Kh = \{\varepsilon h,\, r_{90}h,\, r_{180}h,\, r_{270}h\} = \{h,\, d',\, v,\, \boldsymbol{d}\}$$

   so that $hr_{90} \neq r_{90}h$, but rather $hr_{90} = r_{270}h$. The key here is that when $hK = Kh$, we must say $hk = jh$ for some $j \in K$, instead of $hk = kh$.

5. Consider again the subgroup $K = \{\varepsilon,\, r_{90},\, r_{180},\, r_{270}\}$ of D_4. In this problem, we'll analyze why the set inclusion $dK \cdot vK \subseteq (dv)K$ must hold.

   (a) Find the element $\boxed{?} \in K$ such that $r_{90}v = v\boxed{?}$. **Ans:** $\boxed{?} = r_{270}$.

   Solution. We have $r_{90}v = vr_{270}$, so that $\boxed{?} = r_{270}$.

   (b) Let $\alpha\beta \in dK \cdot vK$, where $\alpha = dr_{90} \in dK$ and $\beta = vr_{180} \in vK$. Using only your result from part (a), *and without looking at the D_4 table again*, explain why $\alpha\beta \in (dv)K$.

   Hint: The argument here should be similar to Problem #3(a). ← But how is it different?

   Solution. We have

   $$\alpha\beta = (dr_{90})(vr_{180}) = d\boldsymbol{(r_{90}v)}r_{180} = d\boldsymbol{(vr_{270})}r_{180} = (dv)(r_{270}r_{180}) \in (dv)K.$$

(c) Explain why $dK \cdot vK \subseteq (dv)K$. Your work from Problem #4 should help.

PROOF. Let $\alpha\beta \in dK \cdot vK$, where $\alpha = dj \in dK$ and $\beta = vk \in vK$ for some $j, k \in K$. Since $Kv = vK$, we have $jv = v\ell$ for some $\ell \in K$. Therefore,

$$\alpha\beta = (dj)(vk) = d\boldsymbol{(jv)}k = d\boldsymbol{(v\ell)}k = (dv)(\ell k) \in (dv)K.$$

Hence $dK \cdot vK \subseteq (dv)K$, as desired. ■

6. **(If you were comfortable with Problem #5, feel free to skip this one!)**

Consider yet again the subgroup $K = \{\varepsilon, r_{90}, r_{180}, r_{270}\}$ of D_4. In this problem, we'll analyze why the set inclusion $hK \cdot d'K \subseteq (hd')K$ must hold.

(a) Find the element $\boxed{?} \in K$ such that $r_{270}d' = d'\boxed{?}$. **Ans:** $\boxed{?} = r_{90}$.

(b) Let $\alpha\beta \in hK \cdot d'K$, where $\alpha = hr_{270} \in hK$ and $\beta = d'r_{90} \in d'K$. Using only your result from part (a), explain why $\alpha\beta \in (hd')K$. Don't look at the D_4 table again!

(c) Explain why $hK \cdot d'K \subseteq (hd')K$. Your work from Problem #4 should help.

Definition. Let H be a subgroup of a group G. Then H is called a *normal subgroup* of G if $gH = Hg$ for all $g \in G$.

- In other words, all left and right cosets of H are equal.
- We often say, "H is normal in G."

---

7. Suppose H is a normal subgroup of G, and let $a, b \in G$.

(a) Let $hb \in Hb$ for some h in H. Elizabeth says, "Since $Hb = bH$, we must have $hb = bh$." How would you correct her claim?

Solution. Coset equality does not imply element-by-element equality. Instead, we have $hb = bj$ for some $j \in H$.

(b) Prove that $aH \cdot bH \subseteq (ab)H$. You may *not* assume that G is commutative.

Hint: $ah \cdot bk = a(hb)k = \cdots$

PROOF. Suppose $x \in aH \cdot bH$ so that $x = ah \cdot bk$ for some $h, k \in H$. Since H is a normal subgroup, we have $bH = Hb$. And since $hb \in Hb$, we have $hb \in bH$ so that $hb = bj$ for some $j \in H$. Therefore,

$$x = ah \cdot bk = a\boldsymbol{(hb)}k = a\boldsymbol{(bj)}k = ab \cdot jk$$

where $jk \in H$. Thus, $x \in (ab)H$ so that $aH \cdot bH \subseteq (ab)H$. ■

8. (a) Explain why the subgroup $H = \{\varepsilon, v\}$ is *not* normal in D_4.

Solution. We have, for instance, $r_{90}H = \{r_{90}, d\}$ and $Hr_{90} = \{r_{90}, d'\}$. Thus, $r_{90}H \neq r_{90}H$ so that H is *not* a normal subgroup of D_4.

(b) Explain why the subgroup $Z = \{\varepsilon, r_{180}\}$ is normal in D_4.

Hint: Z is the *center* of D_4, which means. . . ?

Solution. For each $g \in G$, we have $gZ = Zg$, because we have element-by-element equality, i.e., $gz = zg$ for all $z \in Z$. Thus, Z is normal in G.

(c) **True or False:** If G is commutative, then every subgroup H is normal in G. ← Explain your reasoning.

Solution. True. If G is commutative, then for each $g \in G$, we have $gH = Hg$, because we have element-by-element equality, i.e., $gh = hg$ for all $h \in H$. Thus, H is normal in G.

9. Let N be a normal subgroup of G and let H be any subgroup of G. Define the set product $NH = \{nh \mid n \in N, h \in H\}$. Prove that NH is a subgroup of G.